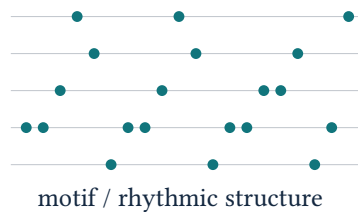
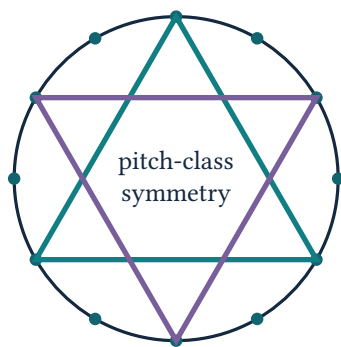


The Symmetry Architecture of *Mercurius*

Counterpoint, mirrored meter, pitch-class groups,
Burnside enumeration and compositional universes



Composer: Christopher Yamba

Mathematical and music-theoretical study prepared collaboratively with ChatGPT

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Evidence: 51-measure MusicXML score of *Mercurius* (1996), from the piano cycle *Planeterna*, written while the composer studied composition with Peter Lyne at Kapellsberg, Härnösand. The date is the composer's own account; the analysis concerns the 2026 digital transcription.

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Abstract

We investigate *Merkurius*, a contrapuntal piano composition attributed by its composer to his teenage years, as a structured system of transformations operating at several musical scales. Analysis of its current 51-measure MusicXML transcription establishes an exact 44-measure palindrome of *meter labels*, from measures 5 through 48, centred between measures 26 and 27. A 20-note five-pitch-class cell occurs in two exactly reversed pairs (measures 6/34 and 19/47); the nominal inversion linking the pairs differs at one pitch-class event. A second 20-note four-class family has exact original, inversion and retrograde exemplars (measures 4/17/36). The opening contrapuntal engine preserves a major-sixth pitch-class interval by pairing descending semitones with ascending major sevenths. Three passages (measures 13, 26 and 40) partition a whole-tone collection into complementary augmented triads. We develop a group-theoretic language for these observations; use Burnside’s lemma to enumerate inequivalent pitch permutations, marked rhythms and multivoice patterns; prove a modular period formula for cyclic motifs with periodic duplications; and derive a parameterised compositional design space. We separate literal properties of the source from interpretative hypotheses and abstract extensions. A Lean 4 companion supplies machine-checkable candidate statements, although the build environment did not contain Lean and the file has not been compilation-verified.

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1 The object of study: score, evidence, and method

This study treats the composer’s digitally archived *Mercurius* as a primary musical document, not as an illustration pasted onto a generic combinatorics argument. The analysed source is the [Mercurius MusicXML score on GitHub](#) (repository `cyamba/simple-yamba-music-pipeline`). The retrieved Git blob begins `c6fe237b` (full SHA in the accompanying README); the XML states a MuseScore encoding date of 2026-10-08. The analytical claims below concern *that particular transcription*. An original manuscript should eventually be collated against it, especially where an isolated note breaks an otherwise exact construction.

The score contains 51 numbered measures, a single piano part on two staves, changing time signatures, both simultaneous notes and distinct contrapuntal lines. The extracted measure inventory has 457 pitched upper-staff events and 473 lower-staff events. We use event rather than sounding voice because chords create multiple simultaneous events. Rests, chord continuations, octave notation and enharmonic spellings require careful interpretation.

Definition 1.1 (Encoded note event). A pitched event is a tuple $e = (b, t, s, p, d)$: measure b , onset t measured in sixteenth-note ticks from the measure beginning, staff $s \in \{U, L\}$, MIDI pitch $p \in \mathbb{Z}$ and duration $d \in \mathbb{N}$. For chromatic pitch-class analysis we project $\pi(p) = p \bmod 12 \in \mathbb{Z}_{12}$. For rhythmic symmetry we retain the local meter and onsets; for registral counterpoint we retain the actual p .

The source sometimes uses chords in one staff while the other moves in fast sixteenths. Therefore, an “upper melody” is not always the upper *staff*: notably the 20-note scalar patterns are in the lower staff in measures 4 and 6, and in the upper staff in measures 17 and 19. A line crossing staff or changing register is not thereby a change of musical idea.

Analytical level	Preserved and measured properties
Temporal form	Meter, bar duration, onset positions, rhythmic palindromes
Melodic syntax	Ordered pitch classes, degree permutations, cyclic repetitions, retrograde
Harmonic/counterpoint	Simultaneous interval classes, registral motion, chord partitions
Transformational	Chromatic T/I operations, voice exchange, time reversal, stabilisers
Generative	Scales, permutation length, rhythmic masks, parts, constraints, equivalence

Table 1: Five distinct analytical levels. A symmetry at one level need not be exact at another.

2 The hidden architectural palindrome

2.1 A 44-measure meter reflection

Write μ_b for the time signature of measure b , as a label such as $5/8$ or $4/4$. The score yields an unexpectedly strong formal relation.

Proposition 2.1 (Exact meter palindrome, source-specific). *For every integer $b \in \{5, 6, \dots, 26\}$,*

$$\mu_b = \mu_{53-b}.$$

Thus the 44-measure passage $5, 6, \dots, 48$ has a perfectly palindromic meter sequence about the boundary between measures 26 and 27.

Proof. The source’s successive time signatures in measures 5–26 are

$$\begin{aligned} &4/4, 5/4, 5/8, 7/8, 5/8, 6/8, 7/8, 6/8, 5/8, 4/4, 4/4, 4/4, \\ &5/4, 4/4, 5/4, 5/8, 7/8, 5/8, 6/8, 7/8, 6/8, 5/8. \end{aligned}$$

Measures 27–48 contain this list in reverse order. The paired measurement table and its illustration give the 22 finite checks. This establishes meter symmetry, not necessarily exact reversal of the notes in every corresponding bar. $\square$

Measures 5 to 26 • forwards



Measures 48 to 27 • backwards



Perfect meter-level reflection: $m \leftrightarrow 53 - m$

Figure 1: Each coloured tile displays an actual measure number and meter. The bottom row is read in reverse measure order; colours agree in all 22 columns. Measures 1–4 and 49–51 are outside the proven core.

The core is *not* a claim that all 51 bars form a note-for-note palindrome. Nor is it enough to establish a global auditory retrograde: metre defines a temporal scaffold upon which the composer can place related, displaced, or modified pitch material. Its structural role is nevertheless unusually explicit. A plausible analytical reading is a reflected large form bounded by opening and closing material.

2.2 Texture, voice exchange and formal turning points

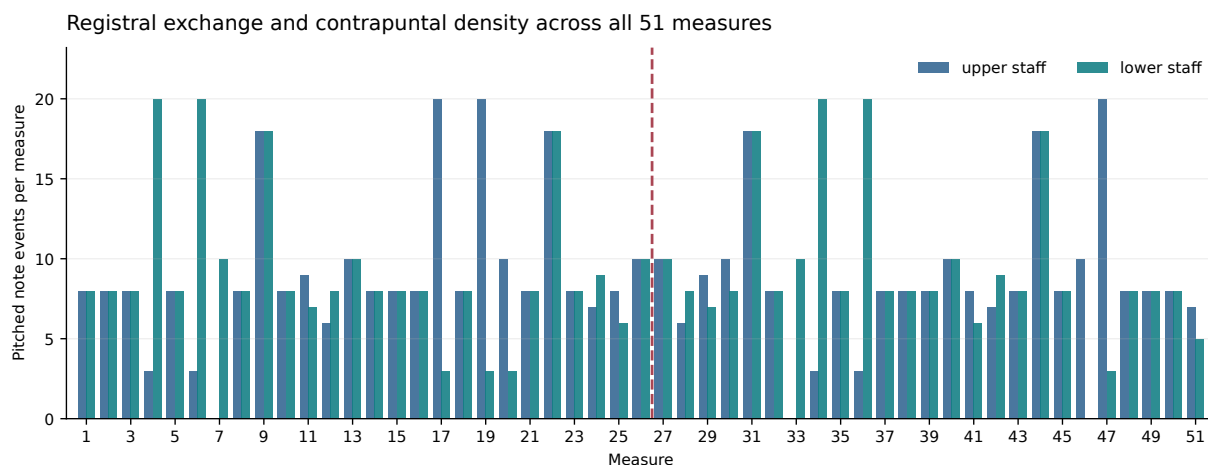


Figure 2: Number of pitched note events in each staff (not the count of independently notated voices). Measures 17–20 and 34–36 exhibit striking changes in which staff bears the rapid activity. The dashed line denotes the meter axis.

The highest local densities occur where a 20-sixteenth motif is presented over a three-note chord; at measures 6 and 34 the lower line is rapid, while at measures 19 and 47 the upper line is rapid. In the score, twelve or more pitched events may also arise from repeated dyads, so high density is not always a single-note flourish. The distinction between *voice distribution* and *abstract thematic identity* is essential when interpreting this piano piece.

3 Two motif families under reflection and reversal

3.1 Operations on pitch-class words

Definition 3.1 (Chromatic inversion and temporal retrograde). For an ordered word $P = (p_1, \dots, p_L) \in \mathbb{Z}_{12}^L$ and $a \in \mathbb{Z}_{12}$, define

$$\begin{aligned} T_a(P) &= (p_1 + a, \dots, p_L + a), & I_a(P) &= (a - p_1, \dots, a - p_L), \\ R(P) &= (p_L, \dots, p_1). \end{aligned}$$

We write $I = I_0$. The operations R and I are commuting involutions: $R^2 = I^2 = e$ and $RI = IR$.

For a generic non-symmetric word these operations generate a four-element orbit governed by the Klein four-group $C_2 \times C_2$. More generally the pitch-class transpositions and inversions T_a, I_a form a dihedral group of order 24, conventionally described as T/I ; temporal reversal commutes with this group and extends it (on ordered words) by a factor C_2 . No assertion that each resulting form must occur in a particular score is implied.

3.2 Five distinct pitch classes: the principal fast cell

In measure 6, the 20-event *lower-staff* pitch-class sequence is

$$\begin{aligned} P_6 &= (8, 8, 11, 4, 10, 6, 8, 11, 11, 4, \\ &\quad 10, 6, 8, 11, 4, 4, 10, 6, 8, 11), \end{aligned}$$

with $C = 0$. The five underlying unique notes are $G^\sharp, B, E, A^\sharp, F^\sharp$, or relative to E Lydian, the degree permutation

$$(3, 5, 1, 4, 2).$$

The score also contains a 20-event upper-staff passage in measure 19, the lower-staff passage in measure 34, and a 20-event upper-staff passage in measure 47.

Proposition 3.2 (Exact pairs, near inversion). *In the extracted digital score,*

$$P_{34} = R(P_6), \quad P_{47} = R(P_{19}).$$

Furthermore P_{19} agrees with $I(P_6)$ at 19 of its 20 event positions, but at position 19 (one-based) the score has pitch class 8 instead of the strictly expected pitch class 4.

Proof. Event-by-event comparison of the source lists shows equalities under reversal. The chromatic inversion check produces exactly one mismatch, at event 19, $E \mapsto G^\sharp$ relative to the strict inversion word. This is a comparison of pitch classes, not an assertion of performance error. $\square$

Four appearances of the 20-event motif; inversion differs at one event

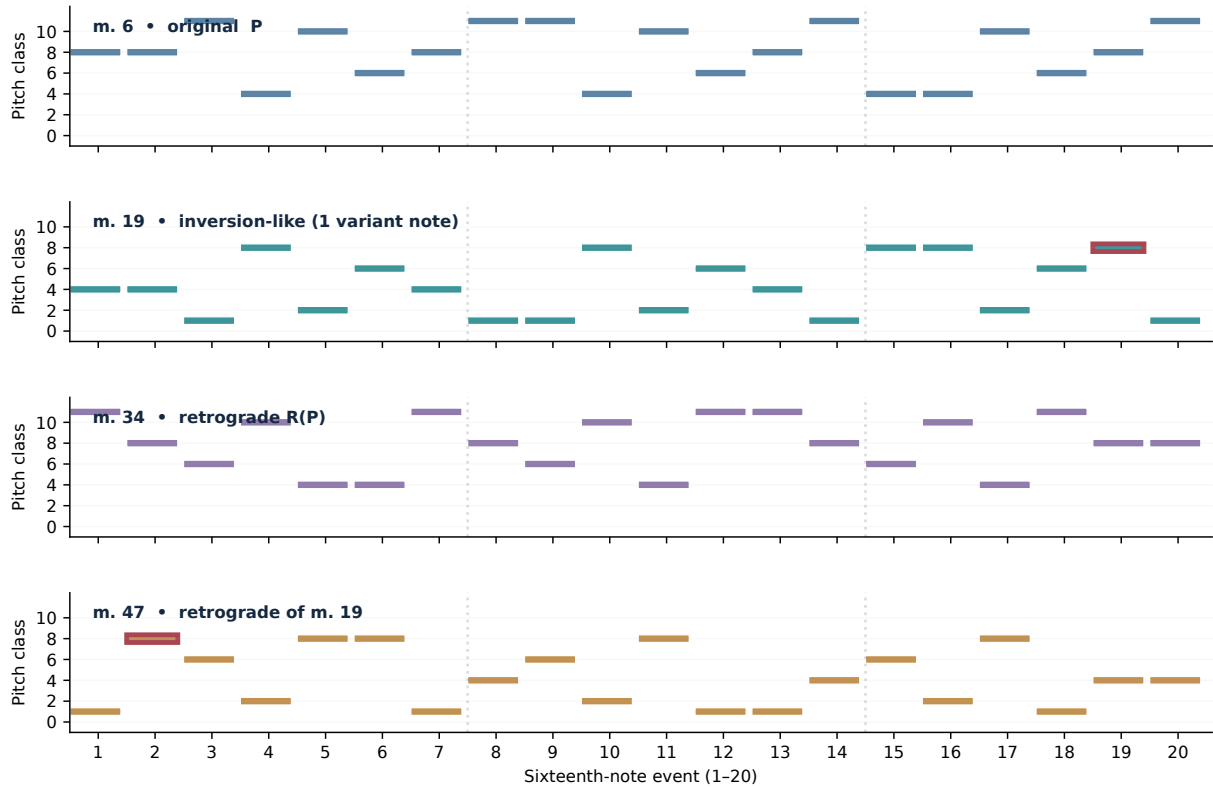


Figure 3: Piano-roll representation of all four 20-event passages (pitch class, not octave). The highlighted red cells mark the corresponding one-event departure from the mathematically exact inversion quartet. The two reverse relationships are exact.

The interesting question is not whether to “repair” this note but whether the divergence is intentional variation, a transcription discrepancy, or an element of a more elaborate compositional constraint. **The score alone cannot choose among these interpretations.** The nearly perfect transformation is itself musically informative.

3.3 Four-note pitch-class family: a second exact system

A different cell (measure 4, lower staff) uses the four classes $\{G^\sharp, C^\sharp, F^\sharp, B\}$. Its event word is presented in the figure below. The measure-17 *upper-staff* word is exactly $I(P_4)$, and the measure-36 *lower-staff* word is exactly $R(P_4)$.

Proposition 3.3 (Independent exact instances). $P_{17} = I(P_4)$ and $P_{36} = R(P_4)$, with all three words containing 20 pitched events.

A second, four-pitch-class motif with exact inversion and retrograde

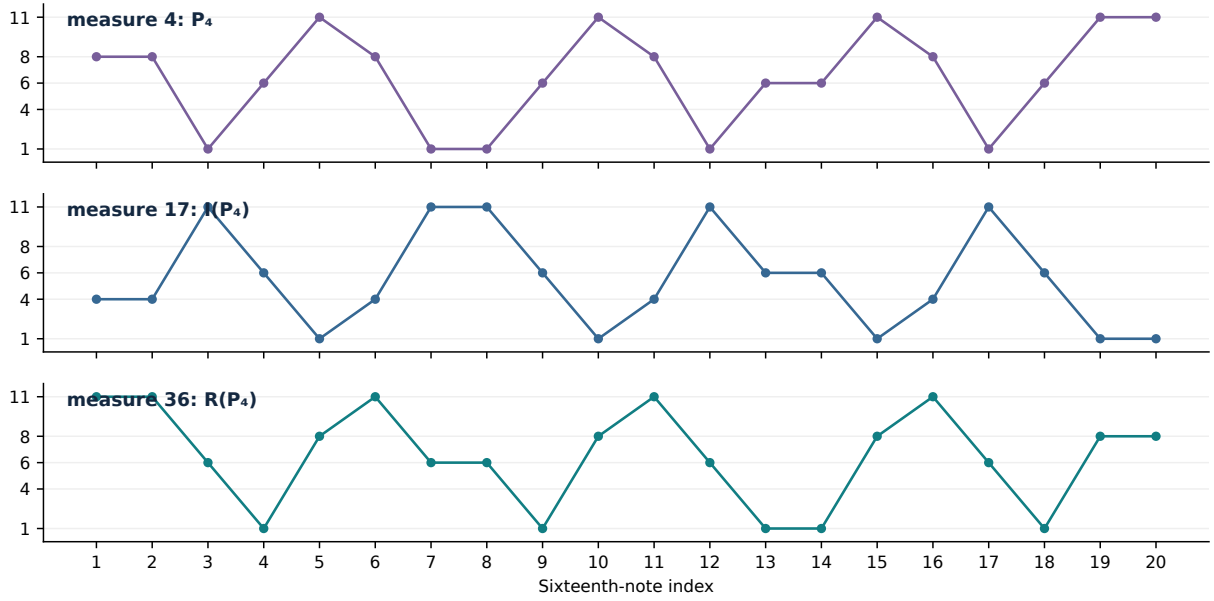


Figure 4: A second family with four distinct pitch classes. Inversion (measure 17) and retrograde (measure 36) are verified exactly, in contrast to the single mutation found in the five-class family.

These two materials have related generative behaviour (few pitch classes, patterned duplication, rigid transformations), but they are not identical seed motifs. A responsible analysis should not conflate them into a single unmodified five-tone row.

4 Mode selection: from E Lydian to E Mixolydian

Definition 4.1 (Mode as a pitch-class collection). With $C = 0$, define

$$L_E = \{1, 3, 4, 6, 8, 10, 11\} \quad (\text{E Lydian}),$$

$$M_E = \{1, 2, 4, 6, 8, 9, 11\} \quad (\text{E Mixolydian}).$$

Theorem 4.2 (Exact modal mapping by chromatic inversion).

$$\boxed{l_0(L_E) = M_E.}$$

Proof. Negating the seven members of L_E modulo 12 gives $\{-1, -3, -4, -6, -8, -10, -11\} = \{11, 9, 8, 6, 4, 2, 1\} = M_E$. $\square$

Chromatic inversion maps the full Lydian collection to Mixolydian

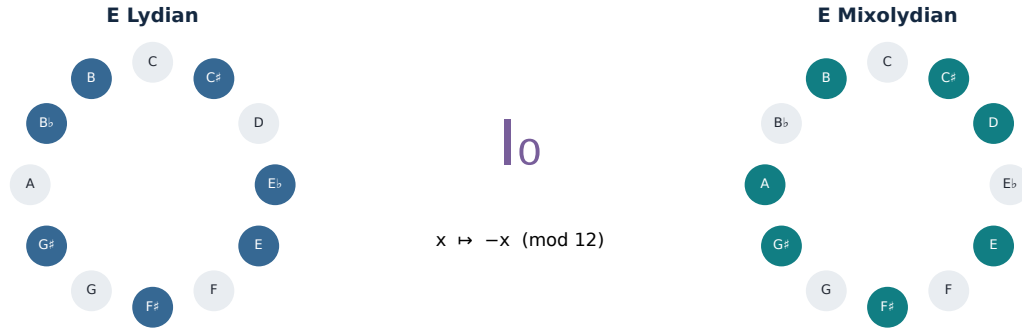


Figure 5: Chromatic inversion of the full seven-note E-Lydian collection produces the complete E-Mixolydian pitch-class collection. The motif itself uses five notes; this is a relation between their compatible seven-note parent collections.

This gives a concrete explanation for the composer’s recollection of a modal shift accompanying a mirror operation. The inversion does not preserve pitch-function degree labels: the original measure-6 seed is (3, 5, 1, 4, 2) in E Lydian, whereas the inverted seed $E, C^\sharp, G^\sharp, D, F^\sharp$ has scale-degree labels (1, 6, 3, 7, 2) in E Mixolydian. That change of degree vocabulary is essential. The term “inversion” here refers to the *twelve-semitone chromatic group*, not merely to reversal of an abstract index set $\{1, \dots, 5\}$. Equating the two would produce incorrect counts and potentially misleading harmonic interpretations.

5 The counterpoint engine: invariant vertical intervals

5.1 A striking construction in the first measure

The first four upper-staff notes are A_7, A_7^b, G_7, G_7^b : MIDI pitches (117, 116, 115, 114). Against them, the lower staff plays C_4, B_4, B_5^b, A_6 : (60, 71, 82, 93). Thus the upper line moves down a semitone while the lower line rises by eleven semitones (a major seventh), step by step. The four upper-minus-lower pitch-class intervals are all 9 (mod 12), i.e., a major sixth modulo octaves. The pattern is repeated within the bar.

Measure 1: contrary motion preserves a major-sixth pitch-class interval

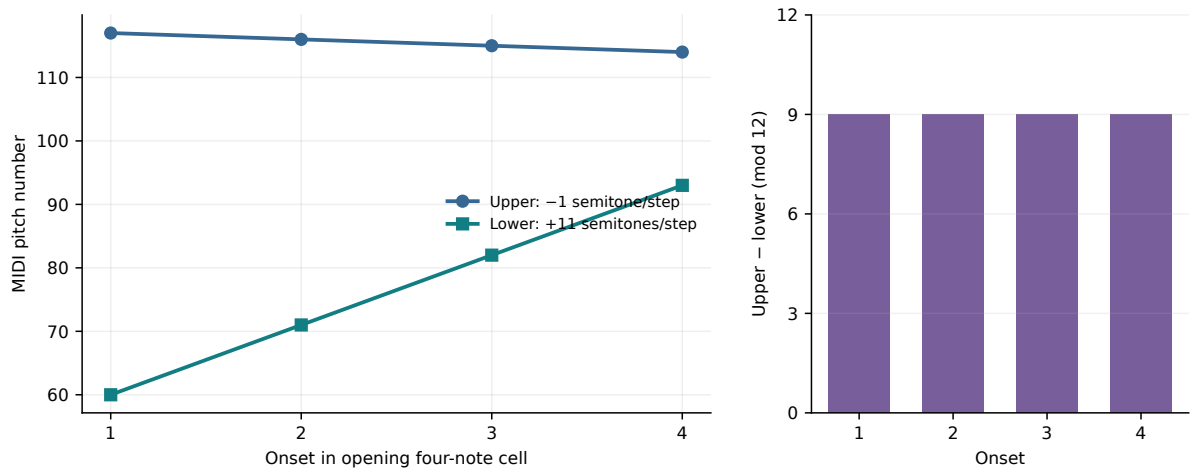


Figure 6: Measures are taken from the first four note attacks of measure 1. Registrational separation narrows by an octave at every step, yet the pitch-class interval remains fixed at 9 semitones.

Theorem 5.1 (Complementary-motion interval invariant). *Let simultaneous voices be integers $u_t = u_0 - at$ and*

$\ell_t = \ell_0 + (12k - a)t$, where $a, k \in \mathbb{Z}$. Then their vertical interval modulo 12 is constant:

$$u_t - \ell_t \equiv u_0 - \ell_0 \pmod{12}.$$

Proof. Subtract the trajectories: $u_t - \ell_t = u_0 - \ell_0 - 12kt$. □

For the opening, $a = 1$ and $k = 1$. A closely related *dual* mechanism occurs in measures 14 and 50: measure 14 descends by eleven semitones in the upper line while the lower rises by one; measure 50 reverses the corresponding gesture. Here the same pitch-class sixth is preserved, although the registral contour is reversed. This is a useful case of *contrapuntal invariance by compensating motions*: horizontal leaps may be extreme while the vertical interval remains stable.

5.2 Intermediary minor sevenths and controlled perturbations

The lower line of measures 3 and 5 mixes large upward motions, including +10 semitones (a minor seventh), with register-changing leaps. The upper line descends predominantly in whole tones. An upper step of -2 against a lower step of $+10$ again produces a vertical change of -12 , so the same invariant principle applies. However, not every onset is invariant. The eight aligned vertical pitch-class differences in measures 3 and 5 are

$$(9, 8, 8, 9, 9, 9, 9, 9).$$

This detail prevents the mistaken claim that every passage obeys a perfectly constant vertical interval. The initial temporary change to class 8 and subsequent return to class 9 can be heard analytically as an interruption and restoration of the interval invariant. It is precisely the kind of local departure that distinguishes *compositional technique* from a rigid mathematical exercise.

5.3 Fifths, fourths and chordal scaffolding

Measures 4, 6, 17, 19, 34, 36 and 47 juxtapose rapid ordered cells with widely spread chord tones, often organised around fifth-related stacks. For instance, the upper sustained chord of measure 6 consists of $E_6, B_6, F_7^\sharp$ (successive perfect fifths); the corresponding measure-19 supporting sonority is $F_3^\sharp, C_4^\sharp, G_4^\sharp$ in the opposite staff. The chord is not necessarily a functional major/minor triad; a chain of fifths may supply an acoustic and intervallic frame while the fast melody carries the transformational argument. This is an instance of *counterpoint between harmonic skeleton and algorithmically ordered surface*.

6 Augmented triads, whole-tone sets, and local harmonic colour

In measure 13, the upper-staff sixteenths cycle through $A^\sharp, F^\sharp, D$; the lower-staff sixteenths cycle through $G^\sharp, E, C$. These are two augmented triads, one chromatic whole step above the other at aligned attacks, with a vertical interval of $2 \pmod{12}$ at all ten event positions. In measure 26 the corresponding collections are $\{E, G^\sharp, C\}$ and $\{D, F^\sharp, A^\sharp\}$, again with interval class 2 at every aligned attack. Measure 40 presents the same whole-tone resources in another ordering, with the same constant interval class. Measure 27 is related but has a four-semitone first aligned difference and two-semitone differences thereafter: this is not an exact vertical-second invariant from its first event.

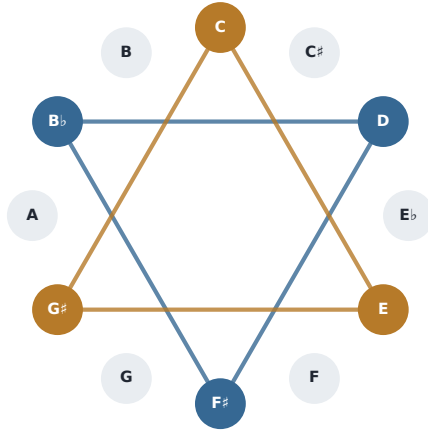
Theorem 6.1 (Complementary augmented-triad decomposition). *For any $r \in \mathbb{Z}_{12}$, let $A_r = \{r, r + 4, r + 8\}$ and $W_r = \{r, r + 2, r + 4, r + 6, r + 8, r + 10\}$. Then*

$$W_r = A_r \dot{\cup} A_{r+2}.$$

Proof. A_r contains the residues $r, r + 4, r + 8$; A_{r+2} contains $r + 2, r + 6, r + 10$. These six are distinct modulo 12, and their union is exactly W_r . □

Augmented triads (0,4,8) and (2,6,10) partition the whole-tone collection

m. 13 • complementary augmented triads



m. 26 • same whole-tone partition

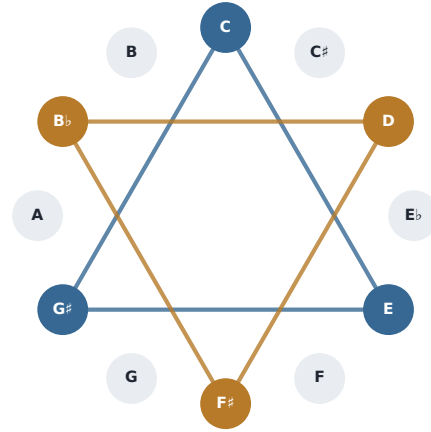


Figure 7: Whole-tone hexachord partitioned into two complementary augmented triads. The bars named here use these very pitch-class resources; the coloured triangles are the triad partitions, not conventional tonic/dominant arrows.

This makes an important musical distinction: the whole-tone sonority is *emergent from complementary contrapuntal parts*. It is not just a chord label assigned after the fact. One can derive its vertical consonance or dissonance profile, temporal phase, arpeggiation and voice exchange independently. In a generator, shifting one arpeggio phase changes the vertical series while leaving both triads unchanged; measure 27 hints at such an additional degree of freedom.

7 Five-note order, rhythmic uniqueness, and a 35-event supercycle

The 20 note-events of measure 6 involve five unique pitch classes. Of their twenty events, three are immediate repetitions introducing one extra note-event each, at event pairs (1, 2), (8, 9) and (15, 16), with one-based indices. Deleting the second event of each pair leaves 17 visits to the repeating cycle

$$P = (G^\sharp, B, E, A^\sharp, F^\sharp) \equiv (3, 5, 1, 4, 2)_{E\text{-Lydian}}.$$

Notice that *seventeen* is not a multiple of five; the 20-event excerpt represents three full underlying cycles plus two extra cyclic visits. This gives a sharper description than “a twenty-note permutation”.

Definition 7.1 (Periodic duplicated-cycle output). Let $P = (p_0, \dots, p_{n-1})$ consist of $n \geq 2$ distinct symbols. For an event spacing $g \geq 2$, define

$$Y_t = P_{f_g(t) \bmod n}, \quad f_g(t) = t - \left\lceil \frac{t}{g} \right\rceil, \quad t \geq 0.$$

The construction repeats the first symbol and then introduces another adjacent duplicate at every g -event boundary.

Theorem 7.2 (Minimal period of a duplicated cyclic motif). *When the seed symbols are all distinct and $g \geq 3$, the infinite event sequence Y has minimal period*

$$T = \frac{ng}{\gcd(n, g-1)}.$$

Proof. One g -event block consumes $g-1$ steps of the underlying n -cycle, because one event is a duplication. As $g \geq 3$ and the seed symbols are pairwise distinct, the adjacent-equality locations identify the block boundaries; hence a period must shift by an integer number h of g -event blocks. The underlying phase returns precisely when $h(g-1) \equiv 0 \pmod{n}$. The least positive h satisfying this congruence is $n/\gcd(n, g-1)$, giving $T = gh$. $\square$

With $n = 5$ and $g = 7$, $\gcd(5, 6) = 1$ and $T = 35$. The finite measure-6 passage is an excerpt of this theoretical 35-event extension, rather than a 35-event phrase actually present in that measure.

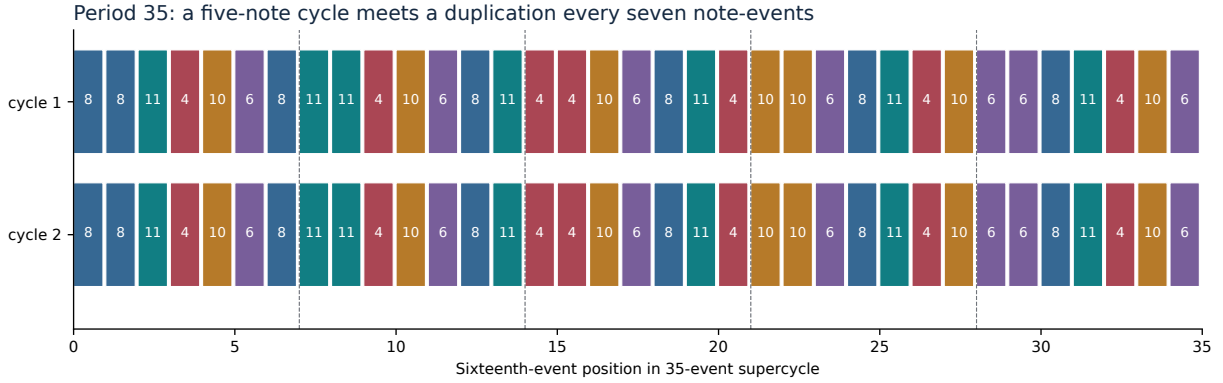


Figure 8: Theoretically continuing measure 6’s five-class cell under its seven-event duplication rule produces a 35-event supercycle. Labels are pitch classes; the second row repeats the first exactly. This extension is mathematical, not an additional claim about the written score.

This construction generalises readily to n pitch classes, a duplication gap g chosen independently of meter, and multiple marked-event patterns. The *cross-period* T is potentially more compositionally fertile than either the pitch cycle or the rhythmic pattern in isolation.

8 Counting with groups, stabilisers and Burnside’s lemma

8.1 The abstract diatonic model and a complete formula

Let X_n be the set of all permutations of the labels $0, 1, \dots, n-1$, so $|X_n| = n!$. We define an *abstract* index reflection $J_n(j) = n-1-j$, which is *not* the same transformation as chromatic inversion modulo 12 in a concrete heptatonic scale. Let I apply J_n to every entry and R reverse the order. The four transformations commute and act on X_n . Under the $C_2 \times C_2$ group G , Burnside’s lemma reads

$$|X_n/G| = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|.$$

Theorem 8.1 (Abstract permutation orbit formula). *For $n \geq 2$, the number of equivalence classes of n distinct diatonic labels under inversion and retrograde is*

$$B(n) = \frac{n! + 2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor!}{4}.$$

Proof. The identity fixes $n!$ permutations. Retrograde alone fixes none, since a permutation with all distinct entries cannot read identically forwards and backwards for $n \geq 2$. Inversion alone fixes none because not every label is a fixed point of J_n . A permutation is fixed by RI exactly when opposing positions carry J_n -paired labels. There are $k = \lfloor n/2 \rfloor$ disjoint pairs of labels; arrange those pairs in the k paired positions ($k!$ choices), choose the orientation of each pair (2^k choices), and place the unique fixed central label if n is odd. Thus $|\text{Fix}(RI)| = 2^k k!$. Burnside gives the result. $\square$

For $n = 5$, $B(5) = (120 + 8)/4 = \boxed{32}$. More finely, there are eight RI -fixed permutations, making four size-two orbits, with the other 112 permutations forming 28 size-four orbits. *This is the correct abstract five-symbol Burnside count; simply dividing $5!$ by four would be wrong.*

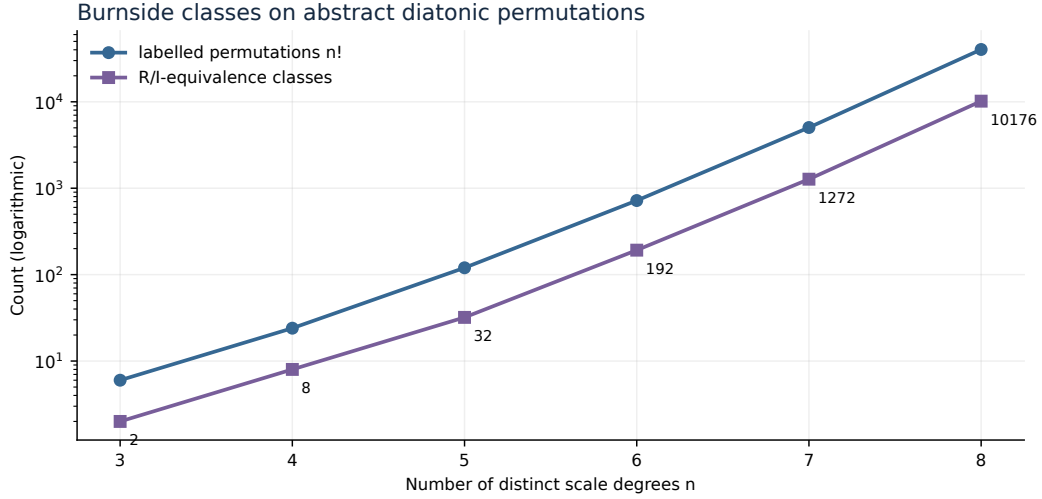


Figure 9: The rapid growth of labelled permutations compared with classes modulo retrograde and index inversion. Counts for $n = 3, \dots, 6$ were additionally checked by exhaustive permutation enumeration.

8.2 The concrete chromatic motif leads to a different count

Let $S = \{4, 6, 8, 10, 11\}$ denote the five pitch classes of measure 6, and $S' = I(S) = \{1, 2, 4, 6, 8\}$. Because $S \neq S'$, a chromatic inversion does not preserve the set of all permutations of *one* of these collections. To obtain a closed domain, take $\mathcal{X} = \text{Perm}(S) \sqcup \text{Perm}(S')$. Then $|\mathcal{X}| = 2 \cdot 5! = 240$. Under e, R, I, RI the only fixed words are those of the identity: R cannot fix a word of distinct notes, while I and RI exchange the two distinct collections. Thus

$$|\mathcal{X}/G| = 240/4 = 60.$$

Both numbers are useful, but describe distinct problems: **32 classes** on one self-reflected abstract degree alphabet; **60 classes** on the two chromatically inversion-related pitch collections. For a *particular generic seed* the four operations produce up to four forms; the written five-class cell has a near-inversion mutation, so its four extracted excerpts should not be called an exact four-member orbit.

8.3 Rhythmic marks: a second Burnside theorem

Take n unique degrees and mark precisely k of their positions for immediate duplication. A phrase object is a pair (P, M) with $P \in X_n$ and $M \subseteq \{0, \dots, n-1\}$, $|M| = k$. Let R reverse both components and let I transform the degree labels but leave the marks fixed.

Theorem 8.2 (Marked permutation phrase classes). *Let $h_{n,k}$ be the coefficient of x^k in*

$$(1+x)^{n \bmod 2} (1+x^2)^{\lfloor n/2 \rfloor}.$$

Then the number of (P, M) -classes under G is

$$B(n, k) = \frac{n! \binom{n}{k} + 2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor! h_{n,k}}{4}.$$

Proof. The identity fixes all $n! \binom{n}{k}$ pairs. Neither R nor I can fix their permutation component. Under RI , the permutation must be one of the $2^{\lfloor n/2 \rfloor} \lfloor n/2 \rfloor!$ fixed permutations from theorem 8.1, and the mask must be palindromic. Each symmetric pair of positions contributes either zero or two marks, a central position (if any) zero or one, giving the generating function stated. Apply Burnside. $\square$

For $n = 5, k = 3$, $h_{5,3} = 2$ and

$$B(5, 3) = \frac{120 \cdot 10 + 8 \cdot 2}{4} = \boxed{304}.$$

Scope warning: this counts five-position marked seed phrases, not the full 20-event cyclic construction of measure 6. For that construction one must additionally specify the expansion length, rhythm phase, and whether retrograde is allowed to change the repetition schedule. Confusing these sample spaces leads to false “unique piece” counts.

8.4 Counting symmetric meters and duration strings

For length L and q allowed rhythmic symbols, the dihedral group of time rotations and retrogrades acts on q^L words. Burnside's bracelet count is

$$\mathcal{B}_q(L) = \frac{1}{2L} \left[\sum_{d|L} \varphi(d) q^{L/d} + Q(L, q) \right],$$

where

$$Q(L, q) = \begin{cases} Lq^{(L+1)/2}, & L \text{ odd}, \\ \frac{L}{2}(q^{L/2} + q^{L/2+1}), & L \text{ even}. \end{cases}$$

Proof. A rotation by r fixes $q^{\gcd(L,r)}$ words; group rotations by cycle length to obtain the totient sum. Reflections of an odd polygon each fix $(L+1)/2$ positions; for even L half of the reflections fix $L/2$ independent positions and half fix $L/2 + 1$. Divide their total fixed points by $2L$. $\square$

This is an abstract extension: the score's *measure-5-to-48* palindrome belongs to the fixed-point set of a particular *reversal* acting on meter words, not necessarily to a cyclic necklace model. The theorem provides a principled way to count possible architectures before composing their notes.

9 A coherent mathematical universe of compositional objects

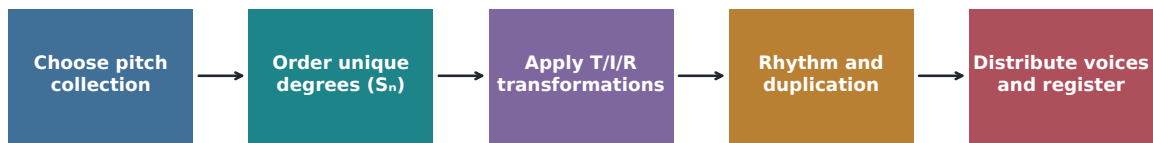
9.1 From notes to transformations to multi-part music

A candidate composition may be represented as

$$\mathcal{C} = (\mathcal{S}, P, \rho, V, H, \mathcal{A}),$$

where $\mathcal{S}$ is a pitch collection, P a melodic word, ρ a rhythm/duration word, V a distribution among voices and registers, H a collection of simultaneous vertical-interval constraints, and $\mathcal{A}$ an architecture of measures and transformations. Different group actions operate on different coordinates: T/I on chromatic pitch classes, R on temporal order, symmetric groups on labellings and voices, and dihedral actions on cyclic rhythms. The “universe” depends on which operations are admitted as equivalences rather than as creative changes.

A generative architecture for families of mathematically related compositions



Add constraints: harmonic interval invariants • symmetrical meter schedules • voice leading • perceptual variety

Figure 10: A constructive reading of the score: interval collections, permutations, transformations, rhythm, and registral distribution are separable parameters subject to musical constraints.

9.2 Families that preserve the Mercurius mechanism but change its sound

1. **Interval-invariant contrary motion.** Choose a and k in theorem 5.1. The opening uses $(-1, +11)$; alternatives include $(-2, +10)$, $(-3, +9)$ or $(-5, +7)$. Preserve the same vertical pitch class while completely changing the melodic contour and register.
2. **Alternative harmonic partitions.** Replace complementary augmented triads with a pair of diminished-seventh tetrachords, complementary trichords of a hexatonic set, or cosets of another subgroup of $\mathbb{Z}_{12}$. Demand fixed vertical intervals on selected onsets and let others vary.

3. **Other scale sizes.** Set $n = 4, 5, 6, 7, 8$ (or 12 for full chromatic serialism). Order all n unique degrees; explore the group orbits, then choose words by shape, tension and balance rather than enumerating them indiscriminately.
4. **Rhythm and duration transformations.** Replace a single repeated pitch every seven events with repeated durations, accents, rests, syncopations, periodic augmentation/diminution or duration-palindrome constraints. Count phrase objects under the group action chosen for the actual rhythmic grammar.
5. **More contrapuntal parts.** For $m = 3, 4, 5$ voices, permit voice permutation, imitation lag, independent T/I actions and vertical-interval matrices; require a bounded crossing count or controlled tension profile. The natural symmetry group can be a product or semidirect product, depending on the allowed independent transformations.
6. **Formal and local symmetries as independent parameters.** Construct an exactly palindromic meter arc but introduce calibrated pitch anomalies, rhythmic phase shifts or voice exchanges. This models the instructive contrast between exact measure-5-to-48 meter symmetry and imperfect exact inversion of one fast cell.

9.3 New definitions for a research programme

Definition 9.1 (Contrapuntal interval-invariance score). For an aligned pair of voices (u_t, ℓ_t) over N attacks, define

$$\text{CIS}_k(u, \ell) = \frac{1}{N} \sum_{t=1}^N \mathbf{1}((u_t - \ell_t) \bmod 12 = k).$$

The score equals one for the opening measure's $k = 9$ pairs and for the aligned whole-tone examples at $k = 2$ in measures 13, 26 and 40. It gives a quantitative measure of how strictly a piece preserves one chosen interval class.

Definition 9.2 (Transformation defect). For equal-length words $P, Q \in \mathbb{Z}_{12}^L$ and a proposed transformation g , define

$$\Delta_g(P, Q) = \#\{j \in \{1, \dots, L\} : g(P)_j \neq Q_j\}.$$

The measure-6-to-19 inversion defect is one; the measure-6-to-34 retrograde defect is zero. This deliberately simple statistic becomes informative when supplemented with melodic interval, register and harmonic severity of each change.

Definition 9.3 (Layered compositional signature). A score may be summarised by

$$\Sigma(\mathcal{C}) = (M, \mathcal{O}, \mathcal{D}, \mathcal{V}, \mathcal{H}),$$

where M records meter-symmetry matches, $\mathcal{O}$ motif orbit representatives and defects, $\mathcal{D}$ rhythmic superperiods, $\mathcal{V}$ voice-distribution maps and $\mathcal{H}$ interval-invariance histograms. A signature is an analytical descriptor, not a complete perceptual measure of originality.

These proposed quantities are not asserted to constitute historically new mathematical discoveries. Their usefulness lies in a transparent, testable application to a real score and in facilitating comparison between compositions.

10 Musical interpretation and critical evaluation

10.1 Counterpoint versus serialism

Merkurius is contrapuntal because independent parts have coordinated, nontrivial simultaneous interval relationships and can exchange foreground/background roles. Its transformational use of permutations, inversion and retrograde is recognisably *serial-like*, but a restricted five-class cell is not a conventional strict twelve-tone row: repeated notes are deliberately inserted, compatible heptatonic modes matter, and chordal/harmonic partitions govern the sound. The work's most explanatory category is arguably *transformational counterpoint with modal and symmetric harmonic constraints*, rather than traditional tonal counterpoint or rigorous dodecaphony.

10.2 What stands out, and what remains to be evaluated

The most distinctive *verified* features of this score are: (i) a long exact meter palindrome surviving changes of surface texture; (ii) a highly explicit major-sixth invariant built from contrary leaps of major sevenths and semitones;

(iii) related but distinct four- and five-class repeated-note transformations; (iv) an exact correspondence between a pitch-class inversion and a change of compatible modal collection; and (v) complementary augmented triads producing whole-tone counterpoint. These features in combination provide a coherent architectural explanation of the work.

Comparisons with Bach, twentieth-century twelve-tone techniques, Messiaen’s modal practice, and post-tonal transformational theory are useful *reference frames*, not evidence that the composition derives from those composers. Lewin’s transformational framework gives a standard mathematical language for T/I actions [2]; Tymoczko’s chord-space perspective provides a different geometric lens on inversion and chord similarity [3]. A defensible judgment of aesthetic quality requires looking beyond formal elegance to audible phrasing, density, contrast, pacing, articulation and performance. This report establishes substantial structural ingenuity but does *not* claim world-historical uniqueness or prove that no other composition uses similar methods.

10.3 Context of the composer’s development

The composer recalls writing *Merkurius* at roughly nineteen years of age, around 1996, in Music Time software, after earlier composition including a partly handwritten piano concerto begun at about sixteen. This provenance makes the analysis doubly valuable: it can identify which formal constraints may have been consciously constructed and which emerged from musical intuition. The digital transcription’s 2026 timestamp must not be mistaken for the original composition date. The next archival step is manuscript comparison, followed by recorded performance or a score-to-audio rendering with careful attention to the very wide registral layout.

Merkurius belongs to *Planetera* (“The Planets”), a series of compositions named after bodies of the solar system, written while the composer studied composition with Peter Lyne at Kapellsberg in Härnösand, Sweden. The series also includes *Charon* for piano, after Pluto’s largest moon, and *Ganymedes* for piano, piccolo and cello, after Jupiter’s moon Ganymede. Of these, *Merkurius* is perhaps the most mathematical. The cycle was conceived before the composer knew Gustav Holst’s orchestral suite *The Planets* (1914–16); the shared subject is a coincidence, not a model.

11 An experimental and formal verification agenda

1. **Source criticism:** photograph or scan any original pages, compare event 19 of measure 19 with the manuscript, and establish whether the unique difference from $I(P_6)$ is intentional.
2. **Exact structural audit:** independently parse the MusicXML to reproduce all 22 meter pairs, all 20-event cells, and aligned interval histograms. The provided CSVs and figure generator give a compact starting point.
3. **Broader harmonic audit:** classify every simultaneous pitch-class set, all successive voice-leading intervals, and chromatic set classes; distinguish sustained chords, implied triads and transient overlaps.
4. **Perceptual study:** make multiple piano recordings/renders with varied registration, tempo and articulation. Compare whether the 44-bar reversal is audible or primarily a composer’s structural symmetry.
5. **Generator implementation:** enumerate orbit representatives rather than all words; choose collections $\mathcal{S}$, a duration grammar, invariant interval targets and voice leading bounds. Generate families with controlled Δ_g anomalies.
6. **Formal mathematics:** mechanise theorems 6.1, 7.2, 8.1 and 8.2 in Lean. Distinguish elementary arithmetic checks from general group-action proofs and retain independent executable tests.

12 Closing perspective

The score suggests not a single trick but a *hierarchy of constrained freedoms*: the clock of the composition can be symmetrically fixed; the notes can be permuted; their pitch classes reflected; phrases reversed; rhythmic duplications phased against the note cycle; and registral motion adjusted to keep selected vertical intervals invariant. What the ear hears as harmonic variety may arise from these discrete operations acting on a surprisingly small seed space. The genuine research opportunity is to move from retrospective analysis to a constructive theory of *families of related, distinguishable compositions*—with enough structure to be mathematically explainable, and enough freedom to remain music.

A Score-grounded measurement ledger

Measures	Source material	Mathematical relation	Qualification
1	upper A, A^b, G, G^b ; lower C, B, B^b, A	vertical class 9 repeated	eighth notes, large octave leaps
3 and 5	descending upper whole tones; lower minor sevenths and leaps	vertical classes $9, 8, 8, 9, 9, 9, 9, 9$	not everywhere invariant
4 / 17 / 36	four-class fast line	$I(P_4), R(P_4)$ exact	20 events; staff changes
6 / 19	five-class fast line	inversion, defect 1	changed event 19/20
6 / 34	five-class fast line	exact retrograde	20 events each
19 / 47	altered five-class fast line	exact retrograde	20 events each
13 / 26 / 40	pairs of augmented triads	$A_r \dot{\cup} A_{r+2} = W_r$	aligned class 2 throughout
27	augmented-triad phase variant	4 then nine 2s	first event differs
14 / 50	broad contrary-motion strands	mirror-like reversal, class 9	exact four-note cells repeated
5–48	meter sequence	$\mu_b = \mu_{53-b}$	exact meter only, not entire music

Table 2: Audit trail of principal assertions. Intervals are pitch-class differences (upper minus lower) unless otherwise specified.

B A computational and Lean companion

The repository includes a `lean/Merkurius.lean` source file, intended for Lean 4 with Mathlib, with statements for involutive operations, the two exact 20-event retrogrades, the one-event inversion discrepancy, the four-class transformation equalities, the E-Lydian/E-Mixolydian pitch-class mapping, augmented-triad partition, interval invariance and numerical Burnside checks. **The Lean source has not yet been compiled against Mathlib**; passing arithmetic `decide/norm_num` goals is different from formalising and verifying the full general Burnside orbit proof. This is a transparent draft verification artefact, not a claim of complete machine-checked certification.

The study’s repository contains the analysed MusicXML (blob `c6fe237b...`) in `score/`, an audit script that re-derives the principal claims directly from it (`src/analyze_musicxml.py`), exhaustive checks of the combinatorial claims (`src/verify_math.py`) and the figure generator (`src/generate_figures.py`); `make check` runs the first two. The full score is not embedded in this PDF; concise music-derived diagrams and transcriptions are used throughout.

References

- [1] Christopher Yamba, *Merkurius*. MusicXML encoding in `cyamba/simple-yamba-music-pipeline`, [GitHub score file](#); source blob `c6fe237b...`. Accessed 17 October 2026.
- [2] David Lewin, *Generalized Musical Intervals and Transformations*. Yale University Press, 1987; Oxford University Press edition, 2007. DOI: [10.1093/acprof:oso/9780195317138.001.0001](#).
- [3] Dmitri Tymoczko, “The Geometry of Musical Chords”, *Science* **313** (2006), 72–74. DOI: [10.1126/science.1126287](#).
- [4] William Burnside, *Theory of Groups of Finite Order*, 2nd ed., 1911 (group-action enumeration commonly known as Burnside’s lemma; also Cauchy–Frobenius lemma).