

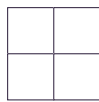
A MATHEMATICAL AND VISUAL TREATISE

Purple Square Atlas

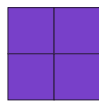
Geometry, symmetry and counting

Christopher Yamba

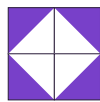
Co-author: GPT-6.1 Sol Extra High



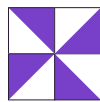
0000



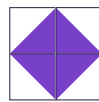
1111



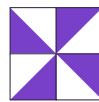
2354



3425



4532



5243

Six oriented square tiles generate a universe of mosaics. Rotating a mosaic moves each tile and turns its internal geometry. We count the resulting classes, construct their symmetric members, and explain why the answer depends on the parity of the board size.

1 296 oriented 2×2 patterns

336 rotation classes

1 The alphabet and the geometric action

A *tile state* is an oriented unit square: white, purple, or divided along a diagonal into a white triangle and a purple triangle. A split tile is named by the right-angle corner of its purple triangle.

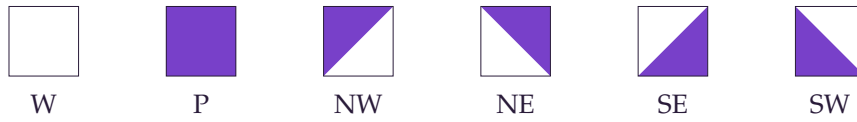


Figure 1: The six states in a fixed frame. Their orientations and colors are distinguished.

Write $T = \{W, P, NW, NE, SE, SW\}$. A clockwise quarter-turn ρ fixes the two solids and cycles the four split states:

$$NW \xrightarrow{\rho} NE \xrightarrow{\rho} SE \xrightarrow{\rho} SW \xrightarrow{\rho} NW.$$

Thus $\rho^4 = \text{id}$, while ρ , ρ^2 and ρ^3 each fix exactly W and P.

For $n \geq 1$, the *board* is the position set $B_n = \{0, 1, \dots, n-1\}^2$. Rows increase downward and columns rightward. A *pattern* is a function $f : B_n \rightarrow T$; the set of patterns is $X_n = T^{B_n}$. Each cell is independent, so $|X_n| = 6^{n^2}$.

The position rotation r and complete-pattern rotation R satisfy

$$r(i, j) = (j, n-1-i), \quad (Rf)(r(p)) = \rho(f(p)). \quad (1)$$

The value at a destination p is therefore $\rho(f(r^{-1}p))$: locate its source, then turn the source tile.

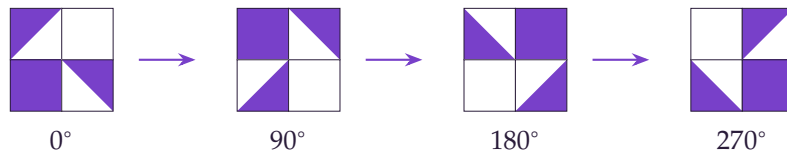


Figure 2: A complete pattern at 0° , 90° , 180° and 270° . Both positions and diagonals turn.

The operations $\{\text{id}, R, R^2, R^3\}$ form the cyclic group C_4 . A *group action* respects composition and the identity operation. The *orbit* of f is the set of its distinct rotated versions; two patterns are equivalent precisely when they lie in the same orbit. We count these orbits. Colors are never exchanged, and no adjacency or inventory constraints are imposed.

2 Position cycles and local consistency

A *position cycle* is the loop obtained by repeatedly applying an operation to one cell until it returns. A quarter-turn cycle has length one, two or four, since $r^4 = \text{id}$.

Any cycle shorter than four is fixed by the half-turn. But

$$r^2(i, j) = (n - 1 - i, n - 1 - j),$$

so a fixed position requires $i = j = (n - 1)/2$. This exists only for odd n ; it is the centre and is also fixed by the quarter-turn. Hence all other quarter-turn cycles have length four. Half-turn cycles away from the centre are pairs.

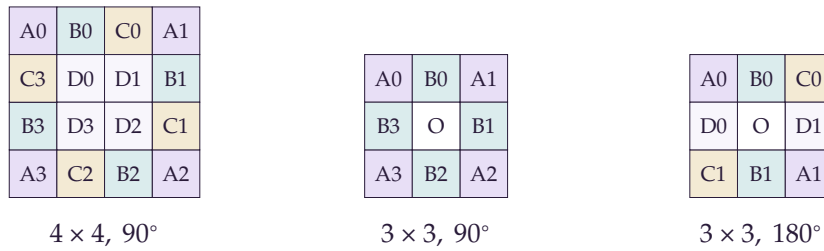


Figure 3: Equal letters identify a cycle; digits give its order. O is the centre. Left: four quarter-turn cycles. Middle: two quarter-turn cycles. Right: four half-turn pairs.

One seed determines a fixed assignment

For a quarter-turn-fixed pattern, values around a four-cycle must be

$$t, \quad \rho t, \quad \rho^2 t, \quad \rho^3 t.$$

Choose t freely; the other values are forced. Returning to the seed requires $\rho^4 t = t$, true for all six states. Recording one seed per cycle and reconstructing by transport are inverse operations, so each cycle contributes exactly six choices.

The centre is stationary, but its tile still turns. It must obey $\rho t = t$, leaving exactly the two solid choices. For a half-turn pair, the values are $t, \rho^2 t$; again all six seeds work. At its centre, $\rho^2 t = t$ leaves two choices.

Let $E_n = |X_n|$, $Q_n = |\text{Fix}(R)|$ and $H_n = |\text{Fix}(R^2)|$, where $\text{Fix}(g) = \{f : gf = f\}$. The inverse operations R and R^3 have the same fixed set. Independence between cycles gives the complete table:

Operation	Even n	Odd n
Identity: E_n	6^{n^2}	6^{n^2}
Quarter-turn: Q_n	$6^{n^2/4}$	$2 \cdot 6^{(n^2-1)/4}$
Half-turn: H_n	$6^{n^2/2}$	$2 \cdot 6^{(n^2-1)/2}$

The fixed object is the complete arrangement. Its individual tiles may move and change orientation while the final arrangement agrees with the original.

3 Why averaging fixed counts counts classes

The required principle is Burnside's lemma, a standard orbit-counting result; see [1, Chapter 7, Lemma 7.2]. We include the proof to make the counting argument self-contained.

Lemma 1 (Burnside). *If a finite group G acts on a finite set X , then*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|. \quad (2)$$

Here X/G is the set of orbits.

Proof. Count the pairs (g, x) for which $gx = x$. Counting by operation gives $\sum_g |\text{Fix}(g)|$; counting by object gives $\sum_x |G_x|$, where the stabilizer $G_x = \{g \in G : gx = x\}$ is the subgroup fixing x .

The map $G \rightarrow Gx, g \mapsto gx$, has fibers of size $|G_x|$: each fiber is a coset of the stabilizer. Therefore $|Gx||G_x| = |G|$. On an orbit of size s , all s objects have stabilizers of size $|G|/s$. That orbit contributes $s(|G|/s) = |G|$ pairs. Every orbit contributes equally; division by $|G|$ proves (2). $\square$

Let $A_n = |X_n/C_4|$. Applying the lemma to the four operations yields

$$A_n = \frac{E_n + Q_n + H_n + Q_n}{4} = \frac{E_n + H_n + 2Q_n}{4}. \quad (3)$$

Theorem 1 (Rotation count). *For every positive integer n ,*

$$A_n = \begin{cases} \frac{6^{n^2} + 6^{n^2/2} + 2 \cdot 6^{n^2/4}}{4}, & n \text{ even}, \\ \frac{6^{n^2} + 2 \cdot 6^{(n^2-1)/2} + 4 \cdot 6^{(n^2-1)/4}}{4}, & n \text{ odd}. \end{cases} \quad (4)$$

Proof. Substitute the fixed counts from Section 2 into (3). The exponents are integral: an even square is divisible by four, and an odd square is one modulo four. $\square$

The group average corrects for the fact that symmetric patterns have fewer distinct views. Division of the total by four is valid only when every orbit has size four, which is not the case here.

4 Worked example: construct and count a 2×2 board

The four cells form one clockwise cycle. We first construct a fixed pattern, then count all such patterns, and finally count the orbits.

1. Choose the top-left seed. Take $t = \text{NW}$. Its purple triangle has its right angle at the top-left corner. Equation (1) forces the top-right tile to be $\rho t = \text{NE}$, the bottom-right to be $\rho^2 t = \text{SE}$, and the bottom-left to be $\rho^3 t = \text{SW}$.

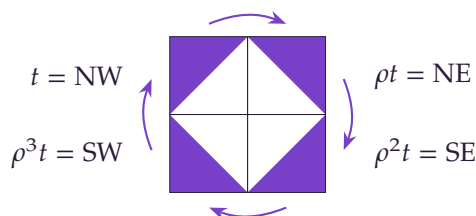


Figure 4: Clockwise transport from a single seed. A further turn returns to $\rho^4 t = t$.

2. Account for every quarter-turn-fixed pattern. Any of the six states may be the seed. Each produces exactly one fixed pattern, and different seeds produce different patterns. Thus $Q_2 = 6$.

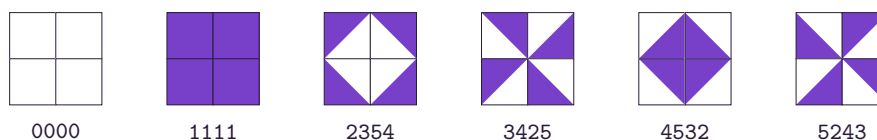


Figure 5: All six quarter-turn-fixed boards. Codes are read top-left, top-right, bottom-left, bottom-right, using $0 = \text{W}$, $1 = \text{P}$, $2 = \text{NW}$, $3 = \text{NE}$, $4 = \text{SE}$, $5 = \text{SW}$.

3. Count the other fixed sets. The identity leaves all four cells free: $E_2 = 6^4 = 1\,296$. A half-turn has two opposite-cell pairs, each determined by one seed: $H_2 = 6^2 = 36$. A three-quarter-turn fixes the same boards as a quarter-turn: $|\text{Fix}(R^3)| = 6$.

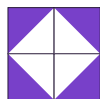
4. Average the four fixed counts. Each fixed set has now been explained geometrically, so Burnside gives

$$A_2 = \frac{1\,296 + 6 + 36 + 6}{4} = 336.$$

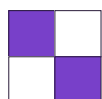
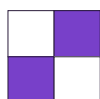
The seed construction establishes the fixed count without enumeration. The software independently visits all $1\,296$ boards and obtains the same result.

5 What the 336 classes look like

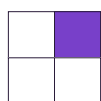
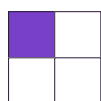
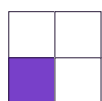
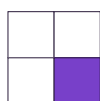
An orbit records distinct views, so repetitions after turning do not create additional members. Since the group has four elements, its orbits can have size one, two or four.



one view; every turn fixes it



two views; a half-turn fixes it



four views; no nonidentity turn fixes it

Figure 6: Examples with one, two and four distinct rotation views. Each row shows one complete orbit.

1. Identify the one-view classes. A board fixed by R is fixed by every turn. These Q_n boards each form a singleton orbit.

2. Identify the two-view classes. A board fixed by R^2 but not R returns after two turns, and has exactly two distinct views. There are $H_n - Q_n$ such boards; pairing them gives $(H_n - Q_n)/2$ orbits.

3. Identify the four-view classes. Every remaining board is not fixed by any nonidentity rotation. The $E_n - H_n$ remaining boards form $(E_n - H_n)/4$ orbits.

Distinct views	General class count	$n = 2$ classes	Boards
1	Q_n	6	6
2	$(H_n - Q_n)/2$	15	30
4	$(E_n - H_n)/4$	315	1 260

There are two different consistency checks: sum classes to count orbits, or weight classes by their sizes to count oriented boards.

$$6 + 15 + 315 = 336, \quad 6 + 2(15) + 4(315) = 1\,296.$$

This decomposition makes the correction in Burnside's average visible. The companion catalogue includes every board and one reproducibly chosen representative of every rotation orbit.

6 Worked example: a centre changes the 3×3 count

The nine positions break into a corner cycle A , an edge-midpoint cycle B , and the centre O . The centre's stationary position does not make an oriented split tile invariant.

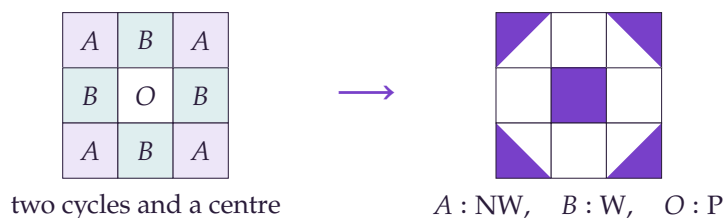


Figure 7: Choose the corner seed NW, the edge seed W, and the centre P. Transport produces the illustrated quarter-turn-fixed board.

1. Choose the two noncentral seeds. The four corners form one clockwise cycle, with values $t, \rho t, \rho^2 t, \rho^3 t$. The four edge midpoints form another. These two seeds are independent, giving $6 \cdot 6$ choices.

2. Impose the centre condition. The centre stays in place while its tile turns. It must satisfy $\rho t = t$, allowing only W or P. Therefore

$$Q_3 = 6 \cdot 6 \cdot 2 = 72.$$

Choosing a split tile at the centre would violate invariance regardless of what is placed in the other eight cells.

3. Repeat for a half-turn. Eight noncentral cells form four opposite pairs. Each pair has six seed choices; the centre must satisfy $\rho^2 t = t$ and again has two. Thus $H_3 = 6^4 \cdot 2 = 2592$. The identity imposes no restrictions, so $E_3 = 6^9 = 10\,077\,696$.

4. Count rotation classes.

$$A_3 = \frac{10\,077\,696 + 2\,592 + 2(72)}{4} = 2\,520\,108.$$

Every positive odd board has precisely the same centre constraint. Increasing the size adds noncentral cycles, while the central factor remains two. For even boards there is no centre at all.

A small boundary check

When $n = 1$, the whole board is its centre. The solids each form one class and the four split orientations form a third class. The odd formula agrees: $(6 + 2 + 2 \cdot 2)/4 = 3$.

7 Reflections introduce a second equivalence

If mirror images are also identified, the acting group is the eight-element symmetry group of the square. We denote it by D_4 ; the subscript counts vertices, while the group's order is eight.

A reflection pairs every cell off its mirror line with another cell. Choose one tile in each pair; the reflected tile is forced. Applying the mirror twice restores every tile, so each pair has six choices. A cell *on* the mirror must instead contain a locally fixed tile.



Figure 8: Tiles fixed by a vertical mirror and by the NW–SE diagonal mirror. Dashed teal lines mark the mirror axes.

1. Count axial mirrors. A horizontal or vertical mirror fixes only the two solid states. An even board has no cells on either axial mirror; an odd board has n such cells. For either axial mirror, the fixed count is

$$V_n = \begin{cases} 6^{n^2/2}, & n \text{ even,} \\ 2^n 6^{(n^2-n)/2}, & n \text{ odd.} \end{cases}$$

2. Count diagonal mirrors. Each diagonal has n fixed cells. On the NW–SE diagonal the allowed states are W, P, NW, SE; on the other diagonal they are W, P, NE, SW. The remaining cells are paired, so either diagonal mirror has fixed count

$$D_n = 4^n 6^{(n^2-n)/2}.$$

3. Average all eight operations. There are two axial and two diagonal mirrors. Hence

$$|X_n/D_4| = \frac{E_n + H_n + 2Q_n + 2V_n + 2D_n}{8}. \quad (5)$$

For $n = 2$, there are two off-axis pairs and no axial fixed cells, so $V_2 = 6^2 = 36$. A diagonal has two fixed cells and one off-diagonal pair, so $D_2 = 4^2 \cdot 6 = 96$. Therefore

$$|X_2/D_4| = \frac{1\,296 + 36 + 12 + 72 + 192}{8} = 201.$$

For $n = 3$, $V_3 = 2^3 6^3 = 1\,728$ and $D_3 = 4^3 6^3 = 13\,824$; substitution gives 1 263 942 classes. These counts answer the question with reflections included, whereas A_n uses rotations.

8 The general principle: transport around a cycle

Let a finite group G act on a finite position set B and a finite state set T . Configurations are functions $f : B \rightarrow T$ with action

$$(g \cdot f)(p) = g \cdot (f(g^{-1} \cdot p)).$$

The action on positions and the action on states may differ; they must both satisfy the group laws. Let $C(g; B)$ be the position cycles of the permutation induced by g .

Theorem 2 (Cycle-product count). *For each $g \in G$, the number of fixed configurations is*

$$|\text{Fix}_{T^B}(g)| = \prod_{C \in C(g; B)} |\text{Fix}_T(g^{|C|})|. \quad (6)$$

Proof. On a cycle of length ℓ , select a start p with value t . Invariance forces $f(g^i \cdot p) = g^i \cdot t$. Returning to p imposes $g^\ell \cdot t = t$. Conversely, each seed satisfying this condition determines exactly one consistent assignment on that cycle. Distinct cycles are independent, so their counts multiply. $\square$

Burnside then averages these products over G . When states have trivial local action, every factor is $|T|$, recovering the usual cycle-counting formula for colorings [1, Theorem 7.5]. Here the local action is part of the geometry and must be retained.

Any quarter-turn alphabet

Suppose $|T| = m$ and its quarter-turn permutation ρ satisfies $\rho^4 = \text{id}$. Set $a = |\text{Fix}(\rho)|$ and $b = |\text{Fix}(\rho^2)|$. The square-board formula becomes

$$|T^{B_n}/C_4| = \begin{cases} \frac{m^{n^2} + m^{n^2/2} + 2m^{n^2/4}}{4}, & n \text{ even,} \\ \frac{m^{n^2} + b m^{(n^2-1)/2} + 2a m^{(n^2-1)/4}}{4}, & n \text{ odd.} \end{cases} \quad (7)$$

Our six-state alphabet has $(m, a, b) = (6, 2, 2)$. Even boards depend only on m : their cycles complete the local rotation. Odd boards also detect a and b through the centre.

This is the natural foundation for the planned *Finite Pattern Lab* sibling: other position sets, alphabets and compatible actions. Inventory or adjacency restrictions need additional methods, because they generally couple cycles that were independent here.

9 Rotating coordinates and verification

Choose a starting position in each four-cycle and write its successive tile values as a_0, a_1, a_2, a_3 . Use coordinates that rotate with the positions:

$$b_i = \rho^{-i} a_i, \quad i \in \mathbb{Z}/4\mathbb{Z}.$$

A complete rotation has $a'_i = \rho a_{i-1}$, so

$$b'_i = \rho^{-i} \rho a_{i-1} = \rho^{-(i-1)} a_{i-1} = b_{i-1}.$$

This is an ordinary cyclic shift. The inverse change $a_i = \rho^i b_i$ recovers every original tile. Thus the change is a bijection commuting with rotation, an *equivariant equivalence*.

For k four-cycles and ε central positions, the configuration space is consequently equivalent to $(T^4)^k \times T^\varepsilon$. A square has $\varepsilon \in \{0, 1\}$ and $k = (n^2 - \varepsilon)/4$. The fixed counts of one four-tuple are 1 296, 6, 36, 6; the counts of one centre are 6, 2, 2, 2. Therefore

$$A = \frac{1296^k 6^\varepsilon + 36^k 2^\varepsilon + 2 \cdot 6^k 2^\varepsilon}{4}. \quad (8)$$

Formal scope and written proofs

The Lean development counts the actual cycle-model orbit quotient and proves (8) for arbitrary natural k and ε , using Mathlib's additive Burnside theorem [2]. The source pins Lean and Mathlib to version 4.19.0. The authorship update does not constitute a new Lean compilation. The grid-to-cycle correspondence, reflection extension and general cycle-product theorem have the written proofs given here and are not additional checked Lean declarations.

Python and JavaScript independently enumerate all 1 296 size-two patterns, verify the geometric group relations, and obtain 336 rotation and 201 dihedral classes. A separate positional-cycle algorithm agrees with the fixed-count formulas for $n = 1, \dots, 20$. Those checks support the implementation; the preceding arguments prove the mathematical statements for every positive size.

References

- [1] Richard P. Stanley. *Topics in Algebraic Combinatorics*, version of 1 February 2013. Chapter 7, especially Lemma 7.2 and Theorem 7.5. <https://math.mit.edu/~rstan/algcomb/algcomb.pdf>.
- [2] The Mathlib community. *Mathlib.GroupTheory.GroupAction.Quotient*, Mathlib v4.19.0. The additive fixed-point/orbit identity used by the accompanying proof. [Pinned source and theorem implementation](#).