

Combinatorial Mathematical Music Theory

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Abstract

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1 Introduction

My goal in this paper is to present a few ways of developing music theory with discrete mathematics. I will try to find mathematical tools valuable when analysing and composing music. I have focused on melodic structure, discrete mathematical interpretation and tools for studying this. Before I introduce some new tools to music theory I will make definitions musical objects and concepts. Mathematics opens a whole new universe of musical possibilities. We can sometimes go through these musical possibilities exhaustively and enumerate them.

What is music?

I will not attempt to give a philosophical explanation of music. However, I will present a graphical way of interpreting music that is effective. Lets say music is sound S over time T . Perhaps some would object because a definition like this incorporates any sound viewed over time and that is not specific enough. I agree here but I do not attempt to be more specific in explaining what music is since this is more of a philosophical topic. However I will be more specific in developing mathematical ways of analysing and generating sound over time when it's ingredients are *pitch* (sound) , *silence* (sound) and *rhythm* (time). In this paper I specify music to be a mapping that maps each "time unit" to a sound where sound is either a set of pitches or sound. This perspective can be investigated with, among many other things, enumerative combinatorics. I will start by defining musical objects and tools for representing various properties of musical objects. Then I will investigate interesting enumerative problems arising from this. As a result new music theory is created, as for example, musical balance theory.

1.1 Musical Sets, Mappings and Musical Metric Spaces

In building a mathematical music theory we will start by dividing “music” into two quantityfiable parts. If music is sound over time. We may think of this as the two parts in question. These two parts can be more specified. Working with combinatorics makes it fruitful to specify the part of time we are interested in as rythm and the type of sound we are interested in is pitch. These are two discrete ways of quantifying Sound and Time. In this section I will present the basic music sets together with various mappings on them and interesting observations about their structure.

1.1.1 Rythm

We shall see that the standard musical notation system for rhythm is abit limited. In this theory we are interested in two types of rhythm.

- 1) The rhythm of a pitch
- 2) The rhythm of silence (The rhythm of a paus) .

When we say rhythm we refer to both 1) and 2)

Figure 1.?

There is a bijection between the set of all rythms R (pitch and paus rythms) and $\mathbb{Q}$.

$$\phi: R \rightarrow \mathbb{Q}, x \mapsto \frac{p}{q}.$$

So in the future we will write down the correspondning rationals $\frac{p}{q}$ when dealing with

rythms. By convention music is considered having an underlying *time signature*, $\frac{p}{q} \in \mathbb{Q}$,

where p is the number of *beats* per bar (one beat correspond to the rythm $\frac{1}{q}$), (*relative*

duration) and where the values for q is, by convention, exclusively $q = 2^k$. Also, by convention, the tempo of the music is the number of beats per minute (actual duration), or

equivalently the solution to the simple equation $\frac{x}{q} = 60$. (60 seconds). The convention

$q = 2^k$ is ofcourse a limitation. Introducing a new way of treating musical rhythm can easily be done by just considering any positive real number q . The reason for $q = 2^k$ can be speculated upon. Perhaps because it easier to keep track of time that is divided into sections of 2 powers rather than any other $q = x^k, x > 2$.

Definition Let us denote the set of all paus-rythms by R_s and the set of all pitch-rythms by R_p . The set of all rythms is $R = R_p \cup R_s$ since this music theory is limited the investigated sound to pitch and paus.

Definition A rhythm *silence (paus) vector* is denoted $r_s = (s_1, \dots, s_{n_1})$, $s_i \in R_s$.

A *rhythm pitch vector* is denoted $r_p = (p_1, \dots, p_{n_2})$. A *rhythm vector* is denoted $r = (r_1, \dots, r_n)$, $n = n_1 + n_2$, $r_i \in R = R_p \cup R_s$.

There's a bijection between R^n and $\mathbb{Q}^n$ that allows us to express rhythm vectors as vectors of rationals.

Since $R^n \sim \mathbb{Q}^n$ we have that R^n can be considered as a vector space over the rational field.

And may also define the scalar product of rhythm vectors r and s as , $r \cdot s = \sum_{i=1}^n r_i s_i$

norm of a rhythm vector $|r| = (\sum_{i=1}^n r_i^2)^{1/2}$. With the given conditions of R^n we have

Among other things that

$$i) |r| \geq 0$$

$$ii) |r| = 0 \Leftrightarrow r = (0, \dots, 0)$$

$$iii) |r - t| \leq |r - s| + |s - t|, r, s, t \in R^n \text{ (follows from } \textit{cauchy-schwarz inequality} \text{)}$$

i), ii) and iii) allows us to regard the set of rhythm vectors R^n as a metric space denoted by $(R^n \times R^n, d)$ where d is the distance function. Since R^n (Let us denote it by $\mathbb{Q}^n$) is an

euclidean space we have $d : \mathbb{Q}^n \times \mathbb{Q}^n \rightarrow \mathbb{R}$ defined by $d(r, s) = |r - s| = \sqrt{\sum_{i=1}^n (r_i - s_i)^2}$,

$r, s \in \mathbb{Q}^n$. This metric gives us an interesting ways of analysing musical rhythm. We may compare rhythms by computing their rhythm vector distance as defined above. We may also define an equivalence relation on R^n by $r \sim s$ if $|r - x| = |s - x|$, where $r, s, x \in R^n$. That is s and r are equivalent in having the same *rhythmical distance* to the rhythm x is one way of interpreting this. Similary we may define equivalence relations by $r \sim s$ if $|r| = |s|$ (rhythm r and s have same distance to the rhythm $0 = (0, \dots, 0)$) and also $(s, r) \sim (u, v)$ if $|s - t| = |u - v|$, $r, s, u, v \in R^n$. If we wish we may define a more “*Hamming*” inspired metric by $(R^n \times R^n, d_h)$ where $d_h : R^n \times R^n \rightarrow \mathbb{R}$ is defined by $d(r, s) = |s - r|$. Similar equivalence relations can be defined on R^n . In ... I will come to why I refer to this metric as “*Hamming*” inspired.

Pitch 1.1.2

The pitch set being build up in subsets (containing *letters*) of cardinality 12 “on top of each other”. In other words, the pitch set have $\mathbb{Z}_{12}$ build into it. We shall define a common equivalence relation, that is the congruence relation on the set of integers. This in a part of music theory called *pitch class analysis* (see [net reference])

The type of music we want to be able to study are those representable within an *n-tone system*. The standard in music today is the *12-tone system* consisting of twelve pitch classes and twelve intervals classes. Therefore it is suitable to introduce something that is in accordance with this standard. We start by partitioning the pitch set into equivalence class sets. The equivalence relation we use is the congruence relation. Given two pitches corresponding to the integers a and b we have $a \equiv b \pmod{n} \Leftrightarrow a \sim b$, for an n -tone system and in particular in a twelve-tone system we get $a \equiv b \pmod{12} \Leftrightarrow a \sim b$. Musicians refer to this by saying two pitches have the same name if the distance between them is an interval that is a multiple of an octave. We get a partitioning of

$P = \{c_k, c\#_k, d_k, d\#_k, e_k, f_k, f\#_k, g_k, g\#_k, a_k, a\#_k, b_k\}, k \in \mathbb{Z}$ in to equivalence classes

$P_c = \{[c], [c\#], [d], [d\#], [e], [f], [f\#], [g], [g\#], [a], [a\#], [b]\}$ where $[x] = \{x_k \in P : k \in \mathbb{Z}\}$.

Definition 1.8: The set of all pitch class sets is denoted $P / \sim = \{[c], [c^\#], [d], [d^\#], [e], [f], [f^\#], [g], [g^\#], [a], [a^\#], [b]\}$

Define a function $\varphi: P / \sim \rightarrow \mathbb{Z}_{12}$ such that

$$\varphi(c) = 0$$

$$\varphi(c^\#) = 1$$

$$\varphi(d) = 2$$

$$\varphi(d^\#) = 3$$

$$\varphi(e) = 4$$

$$\varphi(f) = 5$$

$$\varphi(f^\#) = 6$$

$$\varphi(g) = 7$$

$$\varphi(g^\#) = 8$$

$$\varphi(a) = 9$$

$$\varphi(a^\#) = 10$$

$$\varphi(b) = 11$$

This is an obvious bijection.

Definition 1.1: The set of all pitches is denoted

$$P = \{c_k, c_k^\#, d_k, d_k^\#, e_k, f_k, f_k^\#, g_k, g_k^\#, a_k, a_k^\#, b_k\} = \{c_k, c_k^\#, d_k, d_k^\#, e_k, f_k, f_k^\#, g_k, g_k^\#, a_k, a_k^\#, b_k\} \text{ where } k \in \mathbb{Z}.$$

P is ordered and there is a bijection between P and $\mathbb{Z}$ namely $\phi: P \rightarrow \mathbb{Z}$ defined by

For $k \in \mathbb{Z}$

$$\phi(c_k) = 0 + 12k$$

$$\phi(c_k^\#) = 1 + 12k$$

$$\phi(d_k) = 2 + 12k$$

$$\phi(d_k^\#) = 3 + 12k$$

$$\phi(e_k) = 4 + 12k$$

$$\phi(f_k) = 5 + 12k$$

$$\phi(f_k^\#) = 6 + 12k$$

$$\phi(g_k) = 7 + 12k$$

$$\phi(g_k^\#) = 8 + 12k$$

$$\phi(a_k) = 9 + 12k$$

$$\phi(a_k^\#) = 10 + 12k$$

$$\phi(b_k) = 11 + 12k$$

Note that $\phi(x_k) = \varphi(x) + 12k$ where $x \in P / \sim$ (and $\varphi: P / \sim \rightarrow \mathbb{Z}_{12}$ as defined above)

Define an equivalence relation on $\mathbb{Z}$ such that $a \sim b$ if $a \equiv b \pmod{12}$. We have $\mathbb{Z} / \sim \cong \mathbb{Z}_{12}$.

$$[x] = \{x_k : k \in \mathbb{Z}, x_k \in P\} = \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\} \text{ or equivalently}$$

$$[\varphi(x)]_{12} = \{\phi(x_k) = \varphi(x) + 12k : k \in \mathbb{Z}\} =$$

$$\{\dots, \varphi(x) + (-2) \cdot 12, \varphi(x) + (-1) \cdot 12, \varphi(x) + 0 \cdot 12, \varphi(x) + 1 \cdot 12, \varphi(x) + 2 \cdot 12, \dots\}$$

$\mathbb{Z}/12 = \{0+12\mathbb{Z}, 1+12\mathbb{Z}, \dots, 11+12\mathbb{Z}\} \cong \{[0]_{12}, [1]_{12}, \dots, [11]_{12}\}$. So we can conveniently work in $\mathbb{Z}_{12}$ when dealing with pitch class set $P / \sim$.

Any composition containing pitches will have a pitch set which is a subset of P . We may think of P as a universal set. Thanks to this bijection we may conveniently talk about integers instead of pitches. We will study melodies by considering their pitches disregarding their rhythm.

Figure 1.1

Definition 1.2: A pitch vector of length n is denoted $m = (x_1, \dots, x_n) \in P^n, x_i \in P$.

By the bijection $\phi: P \rightarrow \mathbb{Z}$, we can define a bijective mapping $\phi: P^n \rightarrow \mathbb{Z}^n$,

$$\phi(m) = (\phi(x_1), \dots, \phi(x_n)) = (k_1, \dots, k_n) \in \mathbb{Z}^n.$$

From now on we will mainly look at the corresponding integers vectors of the pitch vectors.

Definition 1.4: A chord-interval between a and b is defined as $\iota_c(a, b) = |b - a|, a, b \in \mathbb{Z}$. Thus defines a “hamming inspired metric” on $\mathbb{Z}$.

Definition A melodic-interval between a and b is defined as $\iota_{m.d.}(a, b) = b - a, a, b \in \mathbb{Z}$

Now we define two sets consisting of intervals.

Definition 1.5: The set of all melodic-intervals is $I_{m.d.} = \{k \in \mathbb{Z} : \iota_{m.d.}(a, b) = k\}, a, b \in \mathbb{Z}$.

We have $I_{m.d.} \sim \mathbb{Z}$.

Definition 1.6: The set of all chord-intervals is $I_c = \{k \in \mathbb{N} : \iota_c(a, b) = k\}, a, b \in \mathbb{Z}$.

We have $I_c \sim \mathbb{N}$.

Definition 1.7: A melodic-interval vector is defined as $i_m = (i_1, \dots, i_n) \in \mathbb{Z}^n$

Definition A chord-interval vector is defined as $i_c = (i_1, \dots, i_n) \in \mathbb{N}^n$

Definition 1.3: A melodic direction-interval between a and b is defined as $\iota_{m.d.}(a, b) = b - a, a, b \in \phi(P) = \mathbb{Z}$. This definition has melodic direction as a consequence.

A melody (see...) with pitch vector $(k_1, \dots, k_n)$ may consecutively go up in pitch, which corresponds to $\iota_{m.d.}(k_i, k_{i+1}) = k_{i+1} - k_i > 0$ or equivalently an ascent. (see ...) or consecutively go down in pitch which correspond to $\iota_{m.d.}(k_i, k_{i+1}) < 0$ or equivalently a descent.

Definition: A melodic direction interval vector of the pitch vector $m = (k_1, k_2, \dots, k_{n-1}, k_n)$ is defined as a function $\phi_{m.d.}: \mathbb{Z}^n \rightarrow \mathbb{Z}^{n-1}$ such that

$$\phi_{m.d.}(m) = (\iota_{m.d.}(k_1, k_2), \dots, \iota_{m.d.}(k_{n-1}, k_n)) = (k_2 - k_1, k_3 - k_2, \dots, k_n - k_{n-1}) = v_{m.d.}$$

Let all k_i in m be distinct, let $v_{m.d.} = (y_1, \dots, y_{n-1}) \in \mathbb{Z}^{n-1}$ define a function $\varphi: \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}_2^{n-1}$ by

$$\phi((y_1, \dots, y_{n-1})) = e_1 \cdots e_{n-1} \text{ such that } e_i = \begin{cases} 1 & \text{if } y_i > 0 \\ 0 & \text{if } y_i < 0 \end{cases}. \text{ Then we see that the direction interval}$$

vector of a pitch vector yields its descent set (see...) of a n -tone series(see...)

Definition: A distance interval vector $v_d \in \mathbb{Z}^{n-1}$ of a pitch vector $(k_1, k_2, \dots, k_{n-1}, k_n) \in \mathbb{Z}^n$ is defined by the function $\phi_d : \mathbb{Z}^n \rightarrow \mathbb{Z}^{n-1}$ such that $\phi_d(m) = (|k_1 - k_2|, |k_2 - k_3|, \dots, |k_{n-1} - k_n|) = v_d$. This is just another way of defining a metric space $(\mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}, d)$ defined by $d : \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1} \rightarrow \mathbb{Z}^{n-1}$ such that $d((k_1, \dots, k_{n-1}), (k_2, \dots, k_n)) = (|k_2 - k_1|, |k_3 - k_2|, \dots, |k_n - k_{n-1}|)$. Again a “hamming inspired” metric.

Definition 1.9: The set of all interval classes of type one is denoted $I_1 / \sim = \{[0]_{12}, [1]_{12}, \dots, [11]_{12}\}$, where $[i]_{12} = \{k : k \equiv i \pmod{12}, k \neq b - a \mid, a, b \in \mathbb{Z}\}$.

Definition 1.11: A type one interval class set of a pitch class set

$P_c = \{p_1, p_2, \dots, p_n, n \leq 12\} \subseteq P / \sim$ is a subset $I_c \subseteq I_1 / \sim$ given by computing all $\binom{n}{2}$ subtractions $|p_i - p_j|, i \neq j$. Let's denote this $I_1 / \sim (P_c / \sim)$.

Example 1: Let $m = (0, 11, 8, 19, 14, 18, 15)$. Then $P_c = \{0, 8, 11, 14, 15, 18, 19\}$,

$P_c / \sim = \{0, 2, 3, 6, 7, 8, 11\}$. $I_1 / \sim (P_c / \sim)$ is decided by the following $\binom{7}{2} = 21$ computations:

0 2 3 6 7 8 11	$ p_j - p_i , i \neq j$
0 2	2
0 3	3
0 6	6
0 7	7
0 8	8
0 11	11
2 3	1
2 6	4
2 7	5
2 11	9
3 6	3
3 7	4
3 8	5
3 11	8
6 7	1
6 8	4
6 11	5
7 8	1
7 11	4
8 11	3

Let's represent this by a type one interval class polynomial $p_{I_{c_1}}(x) = \sum_{k=1}^{11} a_k x^k$. The coefficient

a_i denotes the number of computations $|p_j - p_i| = k, i \neq j$. So $P_c = \{0, 8, 11, 14, 15, 18, 19\}$ results in the polynomial

$$p_{I_{c_1}}(x) = x^2 + 3x^3 + 4x^4 + 3x^5 + x^6 + x^7 + 2x^8 + x^9 + x^{11}$$

Definition 1.10: The set of all interval classes of type two of all interval classes of type one is denoted and defined as $I_2 / \sim (I_1 / \sim) =$

$$\{\{x, y\} : x + y = 12, x, y \in I_1 / \sim\} = \{\{[1], [11]\}, \{[2], [10]\}, \{[3], [9]\}, \{[4], [8]\}, \{[5], [7]\}, \{[6]\}\}.$$

So we can define a type two interval class polynomial by $p_{I_2}(x) = \sum_{k=1}^6 b_k x^k$, where

$b_k = a_k + a_{11-k}$. So the type one interval class polynomial collapses into

$$p_{I_1}(x) = 2x^2 + 4x^3 + 6x^4 + 4x^5 + x^6$$

There are many ways to represent the interval classes of a pitch class vector.

We may represent it as a mapping between *multisets* (see...). The elements of a pitch class vector is a multiset, since there may be repetition entries. So we can define a mapping

between the set of all pitch class multiset $P_M(\mathbb{Z}_{12})$ and the set of all interval class multisets

$P_M(\mathbb{Z}_{12})$. Given the pitch vector $m = (m_1, \dots, m_n)$, $m_i \in \mathbb{Z}$ with corresponding pitch class

multiset $M = \{[0]^{a_0}, [1]^{a_1}, \dots, [11]^{a_{11}}\} \subseteq P_M(\mathbb{Z}_{12})$ by a function $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}_{12}$, $\varphi(m_i) = [k]_{12}$,

$m_i \equiv k \pmod{12}$. (m can be regarded as a word in $S_n(M)$ that is a scattered product of

subwords k^{a_k} , $k \in \mathbb{Z}_{12}$ see...). Define the mapping $\phi: P_M(\mathbb{Z}_{12}) \rightarrow P_M(\mathbb{Z}_{12})$, where $\phi(M) = N$

such that $N = \{[0]^{b_0}, [1]^{b_1}, \dots, [11]^{b_{11}}\}$, where $b_i = \#\{([j]_{12}, [k]_{12}) : |k - j| = i\}$.

In *pitch class analysis* this can be represented by a vector mapping $v: \mathbb{Z}_n \rightarrow \mathbb{N}^{11}$, such that

$$v(m) = (b_0, b_1, \dots, b_{11}), \text{ where } \sum_{k=0}^{11} b_k = \binom{n}{2}. v \text{ is a morphism } v(m_1 * m_2) = v(m_1) + v(m_2). \text{ Where } *$$

denotes concatenation.

Figure 1.? (universum monstrum)

$I_1 / \sim (P' / \sim) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11\}$. $I_1 / \sim (P' / \sim)$ provides not only the information of what intervals can be found in the *melody*, considering any pair of its pitches but also what intervals are not in the *melody*. Knowing these intervals gives the composer the possibility to use them later giving *harmony contrast* (more generally “*musical contrast*” see).

Operations on pitch class sets and melodic vectors

I will go through some of the most common operations on these “musical objects” in music theory and practise. Most of them are frequently used by musicians and composers, but others, like multiplication, are not as common.

Definition: Given a melodic vector $m = (m_1, \dots, m_l) \in \mathbb{Z}^l$, the *musical n-transposition* of m is a mapping $T_n: \mathbb{Z}^l \rightarrow \mathbb{Z}^l$ defined by $T_n(m) = (m_1 + n, \dots, m_l + n)$.

Musical n-transposition corresponds to the vector addition $(m_1, \dots, m_l) + (\underbrace{n, \dots, n}_l)$. Since $\mathbb{Z}^l$ is a vector group under addition, $\mathbb{Z}^l$ is closed under addition (in particular for every vector m and $(\underbrace{n, \dots, n}_n)$) every n-transposition is an element in $\mathbb{Z}^l$.

Figure

Definition: Given a pitch vector $m = (m_1, \dots, m_l)$, the *melodic inversion* of m is

$$I(m) = (-m_1, \dots, -m_l).$$

Melodical inversion of m corresponds to additive invers of m . Since $\mathbb{Z}^l$ is a vector group under addition, there exist an additive inverse for every vector m , thus there exist melodic inversion for every m .

Definition: Define a mapping $\check{\phi}: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ by $\phi(m_i) = m_{n-i+1}$, $1 \leq i \leq n$, $\check{\phi}((m_1, \dots, m_n)) = (\phi(m_1), \dots, \phi(m_n)) = (m_n, \dots, m_1) = m$ is called the *retrograde* of m .

Definition: Define $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}_n$ by $\varphi(x) = [k]_n$, such that $x \equiv k \pmod{n}$

Define a function $\phi: \mathbb{Z}_n \rightarrow [n]$ by $\phi(x) = \begin{cases} k & \text{if } x = [k]_n \\ n & \text{if } x = 0 \end{cases}$.

Given the melodic vector $m = (m_1, \dots, m_n)$, the k -rotation of m is a mapping $R_k: \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ such that $R_k(m) = (m_{\phi \circ \varphi(1+k)}, m_{\phi \circ \varphi(2+k)}, \dots, m_{\phi \circ \varphi(n+k)})$.

Example: Let $m = (m_1, \dots, m_5) = (2, 1, 3, 4, 7)$ then

$$R_3(m) = (m_{\phi \circ \varphi(1+3)}, m_{\phi \circ \varphi(2+3)}, m_{\phi \circ \varphi(3+3)}, m_{\phi \circ \varphi(4+3)}, m_{\phi \circ \varphi(5+3)}) = (m_{\phi(4)}, m_{\phi(0)}, m_{\phi(1)}, m_{\phi(2)}, m_{\phi(3)}) = (m_4, m_5, m_1, m_2, m_3) = (4, 7, 2, 1, 3).$$

We may create compound operations/composite functions of various kind.

We have for example $I \circ \check{\phi} = \phi \circ I$. I and $\check{\phi}$ are idempotent, that is $I^2(m) = I \circ I(m) = m$ and $\check{\phi}^2(m) = \phi \circ \phi(m) = m$. Given that $m \in \mathbb{Z}_n$ and all m_i are distinct R_k , $k = 1$ has order n .

We have $R_a \circ R_b = R_{a+b}$. Thus $R_1^k = \underbrace{R_1 \circ \dots \circ R_1}_k = R_k$ and $R_1^n = \underbrace{R_1 \circ \dots \circ R_1}_n = R_n(m) = m$

In general $R_s^k(m) = R_{sk}(m) = m \Leftrightarrow n \mid sk$. Compound operations involving R_s^k of various kind can be made. For instance $\check{\phi} \circ I \circ R_s^k$.

In chords pitches occur simultaneously in time and therefore there is no melodic direction to consider. A chord can be thought of as a “vertical ascending pitch vector”
Figure

Definition Let $\phi(m) = (k_1, \dots, k_n)$ be a pitch vector such that $k_1 < \dots < k_n$. A *chord vector* c is

defined as a $\phi(m)$ transposed. That is $c = m^t = (k_1, \dots, k_n)^t = \begin{pmatrix} k_1 \\ \vdots \\ \vdots \\ \vdots \\ k_n \end{pmatrix}$.

This will allow us to define series of chords as matrices where columns are chord vectors. Thus we have the mathematical basics for defining *musical polyphony* (multiple layers of melodies) and so the basis of *mathematical musical counterpoint* and *mathematical musical harmony*.

Definition A chord matrix $(k_{i,j})_{n \times n}$ is a matrix with integer entries (corresponding to pitches)

$$\text{Defined by } (k_{i,j})_{n \times n} = \begin{pmatrix} k_{1,1} & k_{1,2} & \dots & k_{1,n} \\ k_{2,1} & \dots & & \\ & & \dots & \\ k_{n,1} & k_{n,2} & & k_{n,n} \end{pmatrix}, \text{ where } k_{1,i} < \dots < k_{n,i}, 1 \leq i \leq n.$$

Observations on chord matrices

It is easy to see that the set of $n \times n$ chord matrices is closed under addition and multiplication but the identity matrix and the inverse of a chord matrix is not a chord matrix so we have

Proposition Let $C_{n \times n}$ denote the set of all $n \times n$ chord matrices. $C_{n \times n}$ is a semi-ring without identity. $\square$

Proposition The only way a the transposed chord matrix $(k_{i,j})_{n \times n}^t$ can be a chord matrix is if all the pitch vectors $(k_{i,j})_{1,n}$ have that $k_{i,1} < \dots < k_{i,n}$. Let us call these matrices *transpose invariant chord matrices*.

Lemma Let $k_{i,j} \neq k_{i,l}, j \neq l, 1 \leq i, j, l \leq n$. Then the number of chord matrices that are not transpose invariant is equal to the number of derangements of n elements.

Lemma Let $(k_{i,j})_{n \times n}$ be a chord matrix such that $k_{i,1} > \dots > k_{i,n}$. Define a mapping

$\phi: C_{n \times n} \rightarrow C_{n \times n}$ such that $\phi((k_{i,j})_{n \times n}) = (k_{i,j-i+1})_{n \times n}$, where $j \in \mathbb{Z}_n = \{1, \dots, n\}$, (0 correspond to index n). Then $(k_{i,j-i+1})_{n \times n}$ is a chord matrix.

Definition Let $(k_{i,j})_{n \times n}$ be a chord matrix. And define a function $f: \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ where $f(i) = j$. A voicing is a pitch vector $v = (k_{f(1),1}, \dots, k_{f(n),n})$.

Proposition The number of voicings of $(k_{i,j})$ is n^n .

Proof By it's definition the voicings corresponds the function f . since f may assume n values for each v_i the result above follows.

We may enumerate voicings according to the twelvefold [1] way given various restriction of f . So we may speak of *injective voicing* and *surjective voicings*.

Musically a voicing means a melody inside a progression of chords. Forexample this could be done by instrumentation, where the composer is assigning a distinct instrument a distinct voicing. That is just one example of musically demonstrating voicings.

Definition A note is defined as a 2-tuple $n = (p, r_p)$, where $p \in P$ and $r_p \in R_p$.

If $p = \emptyset$ then $n = (\emptyset, r_\emptyset) = r_s \in R_s$. We may denote the set of notes by $N = P \times R$,

$R = R_p \cup R_s$. We may distinguish between *silent notes* $n_s = (\emptyset, r_\emptyset) \in N_s$ and *pitch notes*

$n_p = (p, r_p) \in N_p$. $N = N_p \cup N_s$, $N_p \cap N_s = \emptyset$.

Definition A melody is defined as a vector $m = (n_1, \dots, n_k)$, $n_i \in N$

Thus the set of melodies of length k is a set N^k . Since $R \sim \mathbb{Q}$ and $P \sim \mathbb{Z}$ we have

$$N^k = (P \times R)^k \sim (\mathbb{Z} \times \mathbb{Q})^k.$$

Musical contrast

What is musical contrast? How do we define it mathematically? If we ask a musician we are likely to get a description that has to do with how the music is perceived in respect to rythmical complexity, tempo, pitch, chord texture, tonality etc. I will follow this thinking but At a simple level and present a few possible types of musical contrast. Given a piece of music We may create something that sounds different in some way, thus contrasting. What we do is specify the what properties are going to be different. Some musical properties are suitable for this. Properties that are diapole and to witch a value can be assigned, pitch may be high or low, rhythm may contain many rythmical values or few, tempo might be fast or slow, melody might contain a many different notes or just a few, etc. One way of defining contrast is to look at what is not there in one or many ways, in a set-theoretic context this is a matter of complement sets of set. Lets first define musical contrast in a way that it deals with musical sets (def).

Definition: Given a specific musical set A we can define another musical set B such that $B = U \setminus A = \bar{A}$, where U is the universal set. We say that B is the *musical contrast* of A . In other words we define musical contrast in terms of sets as the complement set. Any subset $C \subseteq B$ is a *contrasting set* of A . Let's denote *the set of all contrasting sets* by A_c . In other words A_c is the powerset of $U \setminus A$, that is $A_c = P(\bar{A})$.

Example: consider the pitch class set $A = \{0, 4, 8, 11\}$. U is the set of all pitch classes $\mathbb{Z}_{12}$. The musical contrast of A is $\bar{A} = \mathbb{Z}_{12} \setminus \{0, 4, 8, 11\} = \{1, 2, 3, 5, 6, 7, 9, 10\}$.

We have for the contrasting sets C , that $C \in A_c$ and there are $|A_c| = 2^{|\mathbb{Z}_{12}| - |A|} = 2^{12-4} = 2^8 = 256$ such sets B .

Example: The *pitch class complement set* of A is $\bar{A} = \mathbb{Z}_{12} \setminus A \subseteq P(\mathbb{Z}_{12}) \setminus A$.

Example: One way of achieving *harmony contrast* is to take the *interval class complement* of a interval class set A . That is $I_1 / \sim (P / \sim) \setminus A$.

Other musical constrasts can be created similary on other musical sets

Pitch vectors as permutations of multisets

In this chapter we will investigate another way of representing pitch vectors $m \in \mathbb{Z}^n$. Since a pitch vector may have repetitions of pitches, we are looking at *permutations of multisets*, $m \in S_n(M)$, also refered to as *words*. This approach leads us to representing vectors $m \in \mathbb{Z}^n$ with distinct entries as permutations of sets $m \in S_n$.

Definition: Given a set $A = \{x_1, \dots, x_k\}$, a *multiset* is defined as $B = \{x_1^{a_1}, \dots, x_k^{a_k}\}$ where $\{x_i^{a_i}\} = \{\underbrace{x_i, \dots, x_i}_{a_i}\}, a_i \in \mathbb{N}, 1 \leq i \leq k$. B is a set where repetition is taken into account. It can also

be seen as set A with a function v , (A, v) , where $v: A \rightarrow \mathbb{N}$ is defined by $v(x_i) = a_i$,

$0 \leq a_i \leq n$, If $\#B = n$ we have $\sum_{i=1}^k a_i = n$.

Definition Given a set $A = \{x_1, \dots, x_n\}$, a *submultiset* is denoted as

$C = \{x_1^{b_1}, \dots, x_n^{b_n}\} \subseteq B = \{x_1^{a_1}, \dots, x_n^{a_n}\}$ and defined by the set A together with the function w , (A, w) , where $w: A \rightarrow \mathbb{N}$, such that $w(x_i) = b_i \leq a_i = v(x_i), \forall x_i \in A$.

Proposition: Consider the set $A = \{x_1, \dots, x_n\}$. And the multiset (A, v) , with $v(x_i) = a_i$

multiset $B = \{x_1^{a_1}, \dots, x_k^{a_k}\}$.

The number of submultisets of B is $\prod_{i=1}^n (v(x_i) + 1)$

Proof: Choosing a distinct submultiset C with function w corresponds to choosing a distinct value for u for every x_i in A, between 0 and $w(x_i)$, by the above definition of a submultiset. It

follows that we have $(v(x_1) + 1)(v(x_2) + 1) \cdots (v(x_n) + 1) = \prod_{k=1}^n (v(x_k) + 1)$

values for w and thus equally many submultisets of B.

Definition: Given a pitch vector $m = (m_1, \dots, m_n)$, $m_i \in \mathbb{Z}$, we may have repeated entries.

Thus if we take all m_i 's equal to x_i we have a multiset $B = \{x_1^{a_1}, \dots, x_k^{a_k}\}$. $m = (m_1, \dots, m_n)$ can be regarded as a permutation of this multiset (a word) $S_n(M)$. We say that B is the *pitch multiset of m*, $A = \{x_1, \dots, x_k\}$ is the *pitch set of m* and C is a *pitch sub multiset of m* if $C \subseteq B$.

Definition

The *pitch class set of m*, *pitch class multiset of m* and the *pitch class sub multiset of m*, is defined by mapping the pitches to their pitch classes by the function $\phi: \mathbb{Z} \rightarrow \mathbb{Z}_n$, where $\phi(x_i) = [y_i]_{12}$, such that $x_i \equiv y_i \pmod{12}$.

Let's investigate enumerative aspects of pitch vectors and their pitch multisets.

Consider a pitch vector $m = (m_1, \dots, m_k) \in \mathbb{Z}^k$, or equivalently the word $m \in S_k(B)$,

where $B = \{x_1^{a_1}, \dots, x_n^{a_n}\}$, $\#B = \sum_{i=1}^n a_i = k$. B is called a *k-multiset*.

Given $A = \{x_1, \dots, x_n\}$, $\left(\begin{matrix} n \\ k \end{matrix} \right)$ denote the number of k-multisets B.

Proposition: $\left(\begin{matrix} n \\ k \end{matrix} \right) = \binom{n-1+k}{k} = \binom{n-1+k}{n-1}$.

Proof: Consider the set $A = \{x_1, \dots, x_n\}$ together with a function $v: A \rightarrow \mathbb{N}$, such that

$v(x_i) = a_i$. (A, v) defines the k-multiset $B = \{x_1^{a_1}, \dots, x_n^{a_n}\}$ where

$\sum_{i=1}^n v(x_i) = a_1 + \dots + a_n = k, a_i \in \mathbb{N}, 0 \leq a_i \leq k$. The number of k-multisets is the same as the

number of non-negative integer solutions to the equation $a_1 + \dots + a_n = k$, $a_i \in \mathbb{N}$. It is easy to see that this is in one-to-one corresponding with the number of ways to arrange n-1 |'s and k

•'s. Thus $\left(\begin{matrix} n \\ k \end{matrix} \right) = \binom{n-1+k}{k} = \binom{n-1+k}{n-1}$. $\square$

We can give a generating function where the coefficients are the number of k-multisets of a set of n elements. Let $A = \{x_1, \dots, x_n\}$ and $B = \{x_1^{a_1}, \dots, x_n^{a_n}\}$. Set

$g(x_1, x_2, \dots, x_n) = (1 + x_1 + x_1^2 + \dots)(1 + x_2 + x_2^2 + \dots) \cdots (1 + x_n + x_n^2 + \dots)$. If written out we see that

we get what would correspond to multisets. Let $(1 + x_i + x_i^2 + \dots)$ represent the possible number of i's in B. If we set $x = x_i$, $1 \leq i \leq n$ we get

$$(1 + x_1 + x_1^2 + \dots)(1 + x_2 + x_2^2 + \dots) \cdots (1 + x_n + x_n^2 + \dots) = (1 + x + x^2 + \dots)^n =$$

$$= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{k}x^k + \dots + \binom{n}{n}x^n = \sum_{k=0}^n \binom{n}{k} x^k$$

We have $g(x) = (1 + x + x^2 + \dots)^n = \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k x^k$.

So the number of pitch class k - multisets of a set of n elements is

$$\binom{n}{k} = \binom{n-1+k}{k} = \binom{n-1+k}{n-1} = (-1)^k \binom{-n}{k}.$$

Any pitch vector can be expressed as a permutation of a multiset. If $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$

Then we have $m = m_1 \cdots m_n \in S_n(M)$, where $M = \{m_1, \dots, m_n\}$, m_i, m_j not necessarily distinct. In the future we will work more with pitch vectors represented as words. Alternatovely we may call them *pitch words*.

Corollary: Consider the set $A = \{x_1, \dots, x_n\}$, $n \leq k$ and the multiset $B = \{x_1^{a_1}, \dots, x_n^{a_n}\}$, such that

$$\sum_{i=1}^n a_i = k. \text{ Let } S_k(B) \text{ the set of pitch words of length } k. \#S_k(B) = \sum_{\substack{a_i \geq 0 \\ a_1 + \dots + a_n = k}} \binom{k}{a_1, \dots, a_n}$$

Proof

Given a set of pitches we can build a pitch word of length k by first choosing its pitches, then choosing the order of its pitches. Any of the pitches may be choosen more than once, thus repetition is allowed and the order of the pitches is relevant to the melody so we are infact first choosing multisets then considering permutations on the multiset we've choosen. The numbers of pitch words of length k on a set of n elements is a product of the number of k-multisets and the number of permutations of a k-multiset. Given the set is $A = \{x_1, x_2, \dots, x_n\}$

and $B = \{x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n}\}$, $\sum_{i=1}^n a_i = k$ is our k-multiset.

Let's take a look at the generating function $g(x_1, \dots, x_n) = (x_1 + x_2 + \dots + x_n)^k$

This is similar to the one defined earlier except we consider it "from a pitch word's point of view". The i's parenthesis correponds to choosing the i's pitch of the pitch word, where as the prior generating functions first parenthesis corresponded to "a multisets point of view", choosing the number of occurences of the pitch i. Given

$$g(x_1, \dots, x_n) = (x_1 + x_2 + \dots + x_n)^k = \sum_{a_1 + \dots + a_n = k} x_1^{a_1} \cdots x_n^{a_n}.$$

By the multinomial theorem we have that the coefficients of $x_1^{a_1} \cdots x_n^{a_n}$ is $\binom{k}{a_1, \dots, a_n} =$

$\frac{k!}{a_1! \cdots a_n!}$, which correspond to the number pitch words of length k of pitches from

$A = \{x_1, \dots, x_n\}$. So counting all pitch word is equivalent to summing all coefficients which is the same as evaluating g at $(x_1, \dots, x_n) = (\underbrace{1, \dots, 1}_n)$. Thus

$$g(\underbrace{1, \dots, 1}_n) = n^k = (\underbrace{1 + 1 + \dots + 1}_n)^k = \sum_{\substack{a_i \geq 0 \\ a_1 + \dots + a_n = k}} \frac{k!}{a_1! \dots a_n!} \cdot \square ?$$

If we let $a_1 = \dots = a_n = 1$ we have the case where all pitches are distinct and thus we have

$$\text{The number of pitch words with this property is } \binom{k = \underbrace{1 + 1 + \dots + 1}_n}{\underbrace{1, \dots, 1}_n} = k! = n!.$$

A pitch word like this is called a *n-tone series* which corresponds to a permutation of a n-set of length n in S_n . n-tone series is found in music standard terminology and the most well known musical genre of this kind is *12-tone music*, where the music is build up from a 12-tone series. There are ofcourse $n!$ number of n-tone series.

n-tone series and n-circles

When studying *musical harmony*, *relations between chords* and *progression of chords* the “*circle of fifths*” is relevant. Here I will describe this in terms of n-tone series.

Lets revisit some basic group theory that we will use to establish some results in musical harmony theory.

Definition Let H be a subgroup of G , and let $n \in G$. The set $nH = \{x \in G : x = nh, h \in H\}$ is called the *left coset of H in G determined by n* . The right coset is defined analogically.

In an abelian group all left and right cosets are equal.

Proposition (Lagrange) If H is a subgroup of G . Then $\#H \mid \#G$.

Definition The number left cosets of H in G is called *the index of H in G* , and is denoted by $[G:H]$. $[G:H] = \#G / \#H$. The number of right cosets is defined analogically.

Definition Let $n \in \mathbb{Z}^+$. Euler's φ -function is defined as $\varphi: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$, where

$$\varphi(n) = \#\{a \in \mathbb{Z}^+, a \leq n : (a, n) = 1\}$$

Proposition If the prime factorization of n is $n = \prod_{i=1}^k p_i^{\alpha_i}$, then

$$\varphi(n) = \prod_{i=1}^n \left(1 - \frac{1}{p_i}\right).$$

Definition The set of units of $\mathbb{Z}_n$ is denoted and defined as $\mathbb{Z}_n^\times = \{[a]_n \in \mathbb{Z}_n : (a, n) = 1\}$, where $[a]_n$ is a congruence class.

We have $\#\mathbb{Z}_n^\times = \#\{[a]_n \in \mathbb{Z}_n : (a, n) = 1\} = \varphi(n)$.

Let $G = \mathbb{Z}_{12}$ and let $H = a\mathbb{Z}_{12}, a \in \mathbb{Z}_{12}$. This is the *subgroup generated by a* . Let's denote H by $\langle a \rangle = \{x \in \mathbb{Z}_{12} : x = n \cdot a, \text{ where } n, a \in \mathbb{Z}_{12}\} \subseteq \mathbb{Z}_{12}$. We have that $\#\langle a \rangle = n/(a, n)$.

When $(a, n) = 1$, $\# \langle a \rangle = n$, hence $\langle a \rangle$ contain all the elements of $\mathbb{Z}_n$. We say that a generates $\mathbb{Z}_n$, and $\langle a \rangle$ is a cyclic subgroup of $\mathbb{Z}_n$.

Since $(\mathbb{Z}_n, +)$ is abelian we have $[\langle a \rangle : \mathbb{Z}_n] = [\mathbb{Z}_n : \langle a \rangle] = \frac{\#\mathbb{Z}_n}{\#\langle a \rangle} = \frac{n}{(n/(a, n))} = (a, n)$.

Definition Let $(\mathbb{Z}_n : \langle a \rangle + k)$ denote the set of all cosets of $\mathbb{Z}_n$ on the form $\langle a \rangle + k$, $k \in \mathbb{Z}_n$, $k \notin \langle a \rangle$, $0 \leq k \leq (a, n) - 1$, $a \in \mathbb{Z}_n$. Let $\Sigma_{\langle a \rangle + k}^*$ denote the words $w \in S_{(n/(a, n))}(\langle a \rangle + k)$ over all cosets $\langle a \rangle + k$ in $\mathbb{Z}_n$. Define a function from $\phi : (\mathbb{Z}_n : \langle a \rangle + k) \rightarrow \Sigma_{\langle a \rangle + k}^*$ by

$$\phi(\langle a \rangle + k) = a_1 \cdots a_{(n/(a, n))} = \hat{a}_k, \text{ such that } a_i < a_{i+1}, \text{ where } a_i, a_{i+1} \in \langle a \rangle + k. \text{ We have}$$

$$\phi(\langle a \rangle + k) = \phi(\langle a \rangle) + \phi(k) = \hat{a}_0 + \underset{(n/(a, n))}{(k, \frac{n}{(a, n)}, \frac{n}{(a, n)})} k$$

ϕ is a bijection.

Definition A pitch class word is a word $w = w_1 \cdots w_n$, with letters $w_i \in \mathbb{Z}_{12}$.

Definition Let $n = 12$. Then the word $\hat{a} = a_1 \cdots a_{(n/(a, n))}$ is called an a -circle. We have

$$\phi : (\mathbb{Z}_{12} : \langle a \rangle + k) \rightarrow \Sigma_{12, \langle a \rangle + k}^*, 0 \leq k \leq (a, n) - 1, a \in \mathbb{Z}_{12}, \text{ such that } \phi(\langle a \rangle + k) = a_1 \cdots a_{(12/(a, 12))}$$

$\Sigma_{12, \langle a \rangle + k}^*$ is the set of all a -circles.

The melodical direction interval vector of an a -circle is

$$\phi_{m.d.}(\hat{a}) = (a_2 - a_1, \dots, a_{(n/(a, n))} - a_{(n/(a, n)) - 1}) = \underset{(n/(a, n))}{(a, \frac{n}{(a, n)}, \frac{n}{(a, n)})} a$$

a -circles are said to be *equidistancial* since they are build up of equal intervals and the elements in m are in $\mathbb{Z}_{12}$. The most common n -tone circle is the 7-tone circle, also known as “the circle of fifths”.

Definition Consider the mapping $\phi : (\mathbb{Z}_{12} : \langle a \rangle + k) \rightarrow \Sigma_{12, \langle a \rangle + k}^*$, where

$\phi(\langle a \rangle + k) = a_1 \cdots a_{(n/(a, n))} = \hat{a}_k$. Let $\hat{a}_0$ denote the word mapped by $\phi(\langle a \rangle + 0)$ and $\hat{a}_k$ denote the word mapped by $\phi(\langle a \rangle + k)$. The word $\hat{b} = \hat{a}_{s_0} \cdots \hat{a}_{s_{(12/(a, 12)) - 1}} = b_1 \cdots b_{12} \in S_{12}(\mathbb{Z}_{12})$, $s_i \neq s_j, i \neq j$, $0 \leq s_i \leq (12/(a, 12)) - 1$, $\hat{a}_k \in S_{(n/(a, n))}(\langle a \rangle + k)$ is a word consisting of concatenated a -circles corresponding to the (a, n) cosets created by a . Let's call $\hat{b}$ a *concatenated a -circles*.

Example: Let's study the following 7-circle. We have $\phi(\langle 7 \rangle + k) =$

$$\phi(0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5) + \phi(k) = \underset{n/(a, n)}{(0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5)} + \underset{n/(a, n)}{(k, \frac{n}{(a, n)}, \frac{n}{(a, n)})} k = \hat{a}_k$$

This is “the circle of fifths” represented as a 12-tone series. **Figure**

An appropriate generating function for the number of n -tone series would be $g(x) = \sum_{n=1}^{12} n! x^n$.

The total number of n -tone series is given by $g(1)$. $\mathbb{Z}_{12}$ is a cyclic group. All its elements can

Be generated by elements relatively prime to 12. For example $\langle 5 \rangle = \mathbb{Z}_{12}^\times$.

If we want to find the a-circles with maximal length, we are looking for the elements of $a \in \mathbb{Z}_{12}^\times$, for which $(a, 12) = 1$. We get $\ell(\hat{a}_k) = (12 / (a, 12)) = 12$.

We have $\mathbb{Z}_{12}^\times = \{1, 5, 7, 11\}$, $\varphi(\mathbb{Z}_{12}^\times) = \varphi(12) = 4$. So the a-circle is the 1-circle, 5-circle, 7-circle, and the 11-circle, all of length 12. That is $\phi(\langle 1 \rangle + k) = \hat{1}_k$, $\phi(\langle 5 \rangle + k) = \hat{5}_k$, $\phi(\langle 7 \rangle + k) = \hat{7}_k$ and $\phi(\langle 11 \rangle + k) = \hat{11}_k$. Since $[\mathbb{Z}_{12} : \langle a \rangle, a \in \mathbb{Z}_{12}^\times] = (a, 12) = 1$, there is only one coset of $\langle a \rangle + k$ that is when $k = 0$. One may notice that we have $1+11 = 12$ and $5+7 = 12$ and so $I_2 / \sim (\mathbb{Z}_{12}^\times) = I_2 / \sim (\{1, 5, 7, 11\}) = \{1, 5\}$, (interval class of type two). This equivalence class partitioning of $\mathbb{Z}_{12}^\times$ corresponds to partitioning the a-circles, where $a \in \mathbb{Z}_{12}^\times$, under the equivalence relation where $\hat{a} \sim \hat{b}$ if $\hat{a} = \hat{b}$, where $\hat{a}$ denotes the retrograde of $\hat{a}$.

For example $\hat{7}_0 = (\underset{n/(a,n)}{\underbrace{0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5}}) = \underset{n/(a,n)}{\underbrace{5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7, 0}}$

So the $\hat{7}_0$, “the circle of fifth’s” is essentially no different than $\hat{5}_0$. As where $\hat{1}_0$ and $\hat{11}_0$. They correspond to the ascending and descending “*chromatic scale*”, so they do not convey any interesting 12-tone series. $\mathbb{Z}_{12}^\times$ gives us corresponding equidistancial a-circles $\hat{a}_k$.

We may study a-circles of the other nonunit elements, these are not equidistancial, but *partially equidistancial*. We have the set of non-unit elements

$$\mathbb{Z}_{12} \setminus \mathbb{Z}_{12}^\times = \{1, \dots, 11\} \setminus \{1, 5, 7, 11\} = \{2, 3, 4, 6, 8, 9, 10\}.$$

We want to create 12-tone series that consists the concatenated a-circles, where $a \in \mathbb{Z}_{12} \setminus \mathbb{Z}_{12}^\times = \{2, 3, 4, 6, 8, 9, 10\}$.

Example: Take $a = 2 \in \mathbb{Z}_{12} \setminus \mathbb{Z}_{12}^\times$. We have the subgroup $H = \langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$ and there are $[\mathbb{Z}_{12} : \langle 2 \rangle] = (12, 2) = 2$ cosets, and by the bijection $\phi : (\mathbb{Z}_{12} : \langle a \rangle + k) \rightarrow \Sigma_{12, \langle a \rangle + k}^*$, there are 2 2-circles $\hat{2}_0$ and $\hat{2}_1$. We have $\phi(\langle 2 \rangle + 0) = (0, 2, 4, 6, 8, 10) = \hat{2}_0$ and $\phi(\langle 2 \rangle + 1) = (1, 3, 5, 7, 9, 11) = \hat{2}_1$. The number of a-circles is the number of cosets.

Note that $\langle a \rangle = a\mathbb{Z}_{12}$ and $\mathbb{Z}_{12} / \langle a \rangle \cong \mathbb{Z}_{12/(12,a)}$ by $\psi(k + a\mathbb{Z}_{12}) = \psi(\langle a \rangle + k) = [k]_{(12/(a,12))}$.

So $\Sigma_{12, \langle a \rangle + k}^* \sim \mathbb{Z}_{12/(12,a)}$ such that $[k]_{(12/(a,12))} \in \mathbb{Z}_{12/(12,a)} \leftrightarrow \hat{a}_k \in \Sigma_{12, \langle a \rangle + k}^*$ by $\psi^{-1} \circ \phi$.

Proposition Let $a \in \mathbb{Z}_{12}$. The number concatenated a-circles $\hat{b}$ is $(a, 12)!$

Proof: For all concatenations of a-circles we have $\hat{b} = \hat{a}_{s_0} \cdots \hat{a}_{s_{(a,12)-1}} \in S_{(a,12)}(\{\hat{a}_1, \dots, \hat{a}_{(a,12)}\})$, so $\#S_{(a,12)}(\{\hat{a}_1, \dots, \hat{a}_{(a,12)}\}) = (a, 12)! \quad \square$.

Definition If $a \in \mathbb{Z}_{12}^\times$, then $(a, 12) = 1$ and so there one corresponding concatenation of a-circles $\hat{b} = \hat{a}_0$. This is called a *prime concatenation a-circles*.

Example: Let $k = 3$ and consider $\langle k \rangle = \langle 3 \rangle = \{0, 3, 6, 9\}$. There are $[\langle 3 \rangle : \mathbb{Z}_{12}] = (3, 12) = 3$ cosets which, by the bijection $\phi : (\mathbb{Z}_{12} : \langle a \rangle + k) \rightarrow \sum_{12, \langle a \rangle + k}^*$, correspond to equally many 3-circles. We have $\phi(\langle 3 \rangle + 0) = (0, 3, 6, 9) = \hat{3}_0$, $\phi(\langle 3 \rangle + 1) = (1, 4, 7, 10) = \hat{3}_1$ and $\phi(\langle 3 \rangle + 2) = (2, 5, 8, 11) = \hat{3}_2$. We can concatenate these in $3!$ ways resulting in to the following six 12-tone series: $\hat{\hat{3}}_1 \hat{\hat{c}}_2 \hat{\hat{c}}_3 = (0, 3, 6, 9, 1, 4, 7, 10, 2, 5, 8, 11)$, $\hat{\hat{3}}_1 \hat{\hat{3}}_3 \hat{\hat{3}}_2 = (0, 3, 6, 9)(2, 5, 8, 11)(1, 4, 7, 10)$, $\hat{\hat{3}}_2 \hat{\hat{3}}_3 \hat{\hat{3}}_1 = (1, 4, 7, 10)(0, 3, 6, 9)(2, 5, 8, 11)$, $\hat{\hat{3}}_2 \hat{\hat{3}}_3 \hat{\hat{3}}_1 = (1, 4, 7, 10)(2, 5, 8, 11)(0, 3, 6, 9)$, $\hat{\hat{3}}_3 \hat{\hat{3}}_1 \hat{\hat{3}}_2 = (2, 5, 8, 11)(0, 3, 6, 9)(1, 4, 7, 10)$, $\hat{\hat{3}}_3 \hat{\hat{3}}_2 \hat{\hat{3}}_1 = (2, 5, 8, 11)(1, 4, 7, 10)(0, 3, 6, 9)$.

The concept of a-circles, as most other of the concept dealt with in this paper, can be applied to any *micro-tonality* (division of the octave), simply by studying $\mathbb{Z}_n$ for $n > 12$, or, for that matter, any *macro-tonality*, by studying $\mathbb{Z}_n$ for $n < 12$.

Musical harmony distance series

I will attempt to define *musical harmony distance*. We may see a *main key*, or a *tonal center* c as a pitch from which all other keys have a pitch distance > 0 . The standard is to set $c = c_0$, if no other obvious key is apparent, as the tonal center. But in general $c = x_0$, where $x \in P / \sim$ is the pitch class corresponding to the key signature, or the apparent key, of a composition. I've chosen to do this by introducing a generalized interpretation of "the circle of fifth". Consider "the circle of fifth". That is, the 7-circle $(0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5)$ and its retrograde, the 5-circle $(5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7, 0)$. Apply a function which creates a 12-tone series ascending with respect to a certain function. The music theoretical idea is the relationships between chords. If we start with a "I"-("tonic")-chord C the chord before it is usually a "V7"-("dominant 7")-chord, in this case G. The "V7"-("dominant 7")-chord usually comes after the "IV"-("subdominant") chord, in this case F. If we continue this idea and follow it on the "circle of fifth's" we see that we are alternatively choosing its elements. The idea is that we are choosing the elements with increasing distance from the "tonic", so we are defining an order of musical harmony and implicating relation between different chords. I wish to interpret this idea mathematically.

Figure

Definition: Define a function $\phi_n : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$, $\phi_n(k) = nk(-1)^k$, $k \in \mathbb{Z}_{12}$, $n \in \mathbb{Z}_{12}^\times$.

Since $(a, n) = 1$, this is a bijection.

Definition Define a mapping from the set of all 12-tone series of pitch classes $S_{12}(\mathbb{Z}_{12})$, $(S_{12}(\mathbb{Z}_{12}) \sim S_{12}([12]) = S_{12})$, by $\hat{\phi}_n : S_{12}(\mathbb{Z}_{12}) \rightarrow S_{12}(\mathbb{Z}_{12})$, such that $\hat{\phi}_n(m) = (\phi_n(m_0), \dots, \phi_n(m_{11}))$, $m = (m_0, \dots, m_{11}) \in S_{12}(\mathbb{Z}_{12})$. $\phi(m)$ is a *harmony distance 12-tone series* of m if m is concatenation of a-circles.

Example: If we let $n = 7$, we get $\phi_7(k) = 7k(-1)^k$ and consider the prime concatenation of a-circles $m = \hat{b} = 1_0 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)$, then we get the following harmony distance 12-tone series

$$\hat{\phi}_7(0, 1, 2, \dots, 11) = (\phi_7(0), \phi_7(1), \phi_7(2), \dots, \phi_7(11)) = (0, 5, 2, 3, 4, 1, 6, 11, 8, 9, 10, 7).$$

Let $n = 5$. We get $\phi_5(k) = 5k(-1)^k$, and so

$$\hat{\phi}_5(0,1,2,\dots,11) = (\phi_5(0), \phi_5(1), \phi_5(2), \dots, \phi_5(11)) = (0, 7, 10, 9, 8, 11, 6, 1, 4, 3, 2, 5)$$

since $[5]_{12} = [-7]_{12}$ and $[7]_{12} = [-5]_{12}$. We get

$$\hat{\phi}_7(m) + \hat{\phi}_5(m) = \hat{\phi}_7(m) + \hat{\phi}_{-7}(m) = \hat{\phi}_{-5}(m) + \hat{\phi}_5(m) = (\underset{12}{0, \dots}, 0).$$

Note that if we consider $m_{/0} = m_1 \cdots m_{11} = (1, 2, \dots, 11) \in S_{11}$

$$\hat{\phi}_7(m_{/0}) = (\phi_7(1), \phi_7(2), \dots, \phi_7(11)) = (5, 2, 3, 4, 1, 6, 11, 8, 9, 10, 7) \in S_{11}$$

and $\hat{\phi}_5(m_{/0}) = (\phi_5(1), \phi_5(2), \dots, \phi_5(11)) = (7, 10, 9, 8, 11, 6, 1, 4, 3, 2, 5) \in S_{11}$

So $\hat{\phi}_7(m_{/0}) = \hat{\phi}_5(m_{/0})$.

Proposition Consider a concatenation of a-circles $\hat{b}$.

Then $\hat{\phi}_{[n]_{12}}(\hat{b}) + \hat{\phi}_{[n]_{12}^{-1}}(b) = (\underset{12}{0, \dots}, 0)$ and $\hat{\phi}_{[n]_{12}}(\hat{b}_{/0}) = \hat{\phi}_{[n]_{12}^{-1}}(b_{/0})$.

Proof Consider $\hat{b} = b_0 \cdots b_{11}$. Remember that we are working in $\mathbb{Z}_{12}$, so

$n, k = [n]_{12}, [k]_{12} \in \mathbb{Z}_{12}$. We have $\hat{\phi}_{[n]_{12}}(\hat{b}) = (\phi_n(0), \phi_n(1), \dots, \phi_n(11))$, where

$$\begin{aligned} \phi_n(k) &= n \cdot k \cdot (-1)^k = (-n) \cdot (-k) \cdot (-1)^k = -((-n) \cdot (k) \cdot (-1)^k) = -\phi_{-n}(k) \Leftrightarrow \phi_n(k) + \phi_{-n}(k) = 0 \\ \Rightarrow \hat{\phi}_{[n]_{12}}(\hat{b}) + \hat{\phi}_{[n]_{12}^{-1}}(b) &= (\underset{12}{0, \dots}, 0). \end{aligned}$$

We have $\hat{\phi}_{[n]_{12}}(b_{/0}) = (\phi_n(1), \dots, \phi_n(11)) = (\phi_n(11), \dots, \phi_n(1))$, where

$$\begin{aligned} \phi_n(k) &= n \cdot k \cdot (-1)^k = (-n) \cdot (-k) \cdot (-1)^k = \phi_{-n}(-k) \\ \Rightarrow (\phi_n(11), \dots, \phi_n(1)) &= (\phi_{-n}(-11), \dots, \phi_{-n}(-1)) = (\phi_{-n}(1), \dots, \phi_{-n}(11)) \\ \Rightarrow \hat{\phi}_{[n]_{12}}(\hat{b}_{/0}) &= \hat{\phi}_{[n]_{12}^{-1}}(b_{/0}). \quad \square \end{aligned}$$

We now can put $\hat{b}$ words into a new perspective by making an alternate definition of the prime $\hat{b}$ words (prime concatenation of a-circles).

Definition Define a function $\varphi: \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$, by $\varphi_a(k) = ak \in \mathbb{Z}_{12}$, $a \in \mathbb{Z}_{12}^\times$, $k \in \mathbb{Z}_{12}$.

Then a prime $\hat{b}$ word is a vector $\hat{b} = (\varphi_a(0), \dots, \varphi_a(11))$.

Since $a \in \mathbb{Z}_{12}^\times \Rightarrow -a \in \mathbb{Z}_{12}^\times$, we have that

$$(\varphi_{-a}(0), \varphi_{-a}(1), \dots, \varphi_{-a}(11)) = (\varphi_a(-0), \varphi_a(-1), \dots, \varphi_a(-11)) = (\varphi_a(0), \varphi_a(11), \dots, \varphi_a(1))$$

We can express a harmony distance 12-tone series of a prime word $\hat{b}$ as

$$\begin{aligned} \hat{\phi}_a(\hat{b}) &= (\phi_a(0), \phi_a(1), \dots, \phi_a(10), \phi_a(11)) = (\varphi_a(0), \varphi_a(1), \dots, \varphi_a(10), \varphi_{-a}(11)) = \\ &= (\varphi_a(0), \varphi_a(-1), \dots, \varphi_a(10), \varphi_a(-11)) = (\varphi_a(0), \varphi_a(11), \dots, \varphi_a(10), \varphi_a(1)) = \\ &= (\varphi_a(0), 0, \varphi_a(2), \dots, 0, \varphi_a(10), 0) + (0, \varphi_a(11), 0, \dots, \varphi_a(3), 0, \varphi_a(1)). \end{aligned}$$

Definition Consider a $\hat{b}$ word $\hat{b} = (\phi_a(0), \dots, \phi_a(11)) = b_0 \cdots b_{11}$ and a function $\phi_a: \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$, such that $\phi_a(k) = a \cdot k \cdot (-1)^k \in \mathbb{Z}_{12}$, $a, k \in \mathbb{Z}_{12}$. Define an equivalence relation such that

$(b_i, b_j) \sim (b_s, b_t)$ if $|\phi_a^{-1}(b_j) - \phi_a^{-1}(b_i)| = |\phi_a^{-1}(b_t) - \phi_a^{-1}(b_s)|$. That is, the pair (b_i, b_j) is *harmony distance equivalent* to (b_s, b_t) if they satisfy the distance equality as defined above.

It is easy to see that this is a reflexive, symmetric and transitive relation.

$(\phi_7(1), \phi_5(2), \dots, \phi(10)_7, \phi(11)_5,) (\phi_7(1), \phi_5(2), \dots, \phi(10)_7, \phi(11)_5,) = (0, 7, 5, 2, 10, 9, 3, 4, 8, 11, 1, 6)$.

Note that the 7-circle and the 5-circle intertwined in each other. It's pitches alternates between 7-circle and the 5-circle or equivalently the 7-circle read in its original direction and alternatingly in reverse direction.

Melodic movement and descent statistics

Inclusion exclusion principle

In this section I will study melodic movement of n-tone series. Lets say we have the 4-tone series (1,4,2,3) and (2,4,1,3). Their ...vector is (3,-2,1) and (2,-3,2). When disregarding the "amount of movement" and just considering whether or not the melody is ascending or descending we get that these two melodies have the same type of movement. $a_1 < a_2 > a_3 < a_4$ ($1 < 4 > 2 < 3$) and ($2 < 4 > 1 < 3$). So we can analyse and create music with respect to melodic movement if we develop some tools for studying this. As usual what I call melodic movement is allready a topic in development in mathematics. We will study descent statistics, a part of permutation statistics. It deals with enumerative combinatorics on permutations. I will as before view permutations as a function $\pi: [n] \rightarrow [n]$, $\pi(i) = a_i$ correponding to the word $a_1 a_2 \dots a_n$. Ofcourse originally the melody might be a permutation on a multiset. However I will ofcourse on permutations on ordinary sets and will show some analysing methods for this. Originally the melody might for example be (-11,2,5,21,0,-43) but any such melody would undergo the mapping so that we are dealing with permutations in S_n with $\pi: [n] \rightarrow [n]$, $\pi(i) = a_i$ correponding to the word $a_1 a_2 \dots a_n$ (make a formal def of a function erlier in the text). In this case we have (-11,2,5,21,0,-43) correponding to (2,4,5,6,3,1) and the movement stucture or as we call it descent set of this melody is derived from $(2 < 4 < 5 < 6 > 3 > 1)$. Lets define a descent set.

Definition: Let $\pi \in S_n$, then the descent set of π is $D(\pi) = \{i : a_i > a_{i+1}\}$, $S \subseteq [n-1]$, $\alpha_n(S) = \#\{\pi \in \text{perm}(n) : D(\pi) \subseteq S\}$.

Example: Consider the n-tone series above $\pi = (2,4,5,6,3,1)$, then $D(\pi) = \{4,5\}$.

Now one can ask oneself how many other permutations π in S_6 has got $D(\pi) = \{4,5\}$. How is this computed? Which are the permutations? How are they related via their descent set? This is the permutation statistics we are looking for. Enumeration of the number of permutations with a given descent set. Descent statistics. In finding a formula for descent set statistics we shall find, among other things, the principle of inclusion exclusion very useful. I want to use binary strings as representatives of descent sets therefor I will use a function between sets and binary strings. Let's denote the powerset of S, the set of all subsets $D(\pi) \subseteq S \subseteq [n-1]$, by $P(S)$ and the set of all binary strings of length n-1 by $\mathbb{Z}_2^{n-1}$.

Definition: The function is $\phi: P(D(\pi)) \rightarrow \mathbb{Z}_2^{n-1}$ by $\phi(D(\pi)) = e_1 \dots e_{n-1}$, where

$$e_i = \begin{cases} 1, & \text{if } d_i \in D(\pi) \\ 0 & \text{if } d_i \notin D(\pi) \end{cases}$$

Example: Consider the prior example where $\pi = (2,4,5,6,3,1)$, and set $D(\pi) = \{4,5\}$. In this case we have $D(\pi) \subseteq S \subseteq [5]$ and $\phi(D(\pi)) = 00011$

Lets study the descent statistic further.

Definition: Given $S \subseteq [n-1]$ let $\alpha(S)$ denote the number of permutations $\pi \in S_n$ who's descent set is contained in S. That is $\alpha(S) = \#\{\pi \in S_n : D(\pi) \subseteq S\}$.

Proposition: Let $S = \{s_1, \dots, s_k\} \subseteq [n-1]$. Then $\alpha_n(S) = \binom{n}{s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_k}$

Proof: Consider the permutation $\pi = a_1 a_2 \dots a_n$ with $D(\pi) \subseteq S$, we may first choose

$a_1 < a_2 < \dots < a_{s_1}$ in $\binom{n}{s_1}$ ways. Proceed by choosing $a_{s_1+1} < a_{s_1+2} < \dots < a_{s_2}$ in $\binom{n-s_1}{s_2-s_1}$ ways

etc. From this we get $\alpha_n(S) = \binom{n}{s_1} \binom{n-s_1}{s_2-s_1} \binom{n-s_2}{s_3-s_2} \dots \binom{n-s_k}{n-s_k} = \binom{n}{s, s-s, \dots, n-s}$ as desired.

Now lets define the number of permutations $\pi \in S_n$ with descent set equal to S.

Definition: Given $S \subseteq [n-1]$ denote the above by $\beta(S) = \#\{\pi \in S_n : D(\pi) = S\}$.

It is clear that $\alpha(S) = \sum_{T \subseteq S} \beta(T)$ and this holds the key to using the principle of inclusion

exclusion to get a formula for $\beta(S)$. So lets look further into this principle in this context.

Theorem: Let S be a set with $\#S = n$. Let V be a 2^n - dimensional vector space (over a field k) of all functions $f : P(S) \rightarrow k$. Let $\phi : V \rightarrow V$ be the linear transformation

$\phi f(T) = \sum_{Y \supseteq T} f(Y), \forall T \subseteq S$. Then there exists a ϕ^{-1} such that

$$\phi^{-1} f(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f(Y), \forall T \subseteq S$$

(Proof...)

Think of S as the set of properties that the elements of a given set A may or may not have. For $T \subseteq S$ let $f_=(T)$ be the numbers of objects in A with exactly the properties in T.

Let $f_{\geq}(T)$ be the number of objects in A with at least the properties in T, then

$$f_{\geq}(T) = \sum_{Y \supseteq T} f_=(Y)$$

From Theorem it follows that $f_=(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f_{\geq}(Y)$

We get the dualized version of the Theorem by looking at $f_{\leq}(T)$ instead of $f_{\geq}(T)$.

Let $f_{\leq}(T)$ be the number of objects of A having at most the properties in T, then interchanging $\supseteq$ by $\subseteq$,

$$f_{\leq}(T) = \sum_{Y \subseteq T} f_=(Y)$$

$$f(T) = \sum_{Y \subseteq T} (-1)^{|T-Y|} f_{\leq}(Y)$$

When $|T| = |T'| \Rightarrow f_{\leq}(T) = f_{\leq}(T')$,

$f_{\geq}(T)$ also only depends on $|T|$. Set $|T| = i \Rightarrow a(n-i) = f_{\leq}(T)$ and $b(n-i) = f_{\geq}(T)$.

$$f_{\geq}(T) = \sum_{Y \supseteq T} f_{\leq}(Y) \text{ corresponds to } b(m) = \sum_{i=0}^m \binom{m}{i} a(i), 0 \leq m \leq n. \text{ and}$$

$$a(m) = \sum_{i=0}^m \binom{m}{i} (-1)^{m-i} b(i) \text{ corresponds to } f(T) = \sum_{Y \subseteq T} (-1)^{|T-Y|} f_{\leq}(Y).$$

What we have is the inverse of the $(n+1) \times (n+1)$ matrix with (i,j) -entry $\binom{j}{i}, (0 \leq i, j \leq n)$ that

has (i,j) -entry $(-1)^{j-i} \binom{j}{i}$

Example: (give a matrix general or a specific)

Now we have sufficient tools for computing $\beta_n(S)$. $f(T) = \sum_{Y \subseteq T} (-1)^{|T-Y|} f_{\leq}(Y)$ (state the nr of

the theorem) we get $\beta_n(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_n(T) = \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq k} (-1)^{k-j} \binom{n}{s_{i_1}, s_{i_2}, \dots, s_{i_j}}$, since

$\alpha_n(S) = \binom{n}{s_1, s_2, \dots, s_k}$, $S = \{s_1, s_2, \dots, s_k\} \subseteq [n-1]$. Writing out the sum would require us

to have a way of listing all the subsets of S that is listing all the elements in P(S).

Lets denote every $T \subseteq S$ by T_j^l where $j = |T|$ and l is the index of the subset defined by some function such that the order of the listing of subsets T is established by j and $f(T_j^l)$ so that $T_{j_b}^{l_b} \Leftrightarrow f(T_{j_a}^{l_a}) < f(T_{j_b}^{l_b})$. We have the set of all subsets T of S is $P(S) = \{$

$T_0^1, T_1^1, T_1^2, \dots, T_1^k, T_2^1, T_2^2, \dots, T_2^{\binom{k}{2}}, \dots, T_j^1, T_j^2, \dots, T_j^{\binom{k}{j}}, \dots, T_k^1, T_k^2, \dots, T_k^{\binom{k}{k}} \}$ and

Every T_j^l corresponds to a unique term $\binom{n}{s_{i_1}, s_{i_2}, \dots, s_{i_j}}$ so we get a sum of $2^{|S|} = 2^k$ terms.

$$\beta_n(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_n(T) = \sum_{T_j^l \in P(S)} (-1)^{|S-T_j^l|} \alpha_n(T_j^l) = \sum_{j=k}^{j=0} (-1)^{k-j} \left(\sum_{l=1}^{\binom{k}{j}} \alpha(T_j^l) \right).$$

Example: Given $n-1=8$ and $S=\{1,5\}$ We get

$S = \{s_1=1, s_2=5\} \subseteq \{1, 2, \dots, 8\}$. We have the subsets $T_2 = T_2^{\binom{2}{2}} (=S)$, $T_1 = T_1^1 \cup T_1^{\binom{2}{1}}$ and

$$T_0 = \{\emptyset\}. \text{ Thus } \beta_n(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_n(T) = \sum_{j=k}^{j=0} (-1)^{k-j} \left(\sum_{l=1}^{\binom{k}{j}} \alpha(T_j^l) \right) =$$

$$= (-1)^{2-0} \cdot \binom{8}{1, 5-1, 8-5} + (-1)^{2-1} \left(\binom{8}{1, 8-1} + \binom{8}{5, 8-5} \right) + (-1)^{2-2} \binom{8}{0} = 217. \text{ Ofcourse}$$

$2^{k-2} = 2^2 = 4$ terms. As we mentioned earlier it is possible to find inverse matrix expression for the inclusion exclusion formula. Notice that if we let g be a function defined on

$[k+1] \times [k+1]$ with $i > j \Rightarrow g(i, j) = 0$. The non-zero terms

$A_k = \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq k} (-1)^{k-j} g(0, i_1) g(i_1, i_2) \dots g(i_j, k+1)$ of the expansion of the determinant with

(i, j) -entry $g(i, j+1) \Rightarrow (i, j) \in [0, k+1] \times [0, k+1] = \{0, 1, \dots, k+1\} \times \{0, 1, \dots, k+1\}$. If we set

$g(i, j) = 1/(s_j - s_i)!$, with $s_0 = 0$ and $s_{k+1} = n$. So we have

$$\begin{aligned} A_k &= \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq k} (-1)^{k-j} g(0, i_1) g(i_1, i_2) \dots g(i_j, k+1) \\ &= \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq k} (-1)^{k-j} \frac{1}{(s_{i_1} - 0)!} \frac{1}{(s_{i_2} - s_{i_1})!} \dots \frac{1}{(s_{k+1} - s_{i_j})!} = \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq k} (-1)^{k-j} \frac{1}{(s_{i_1})!} \frac{1}{(s_{i_2} - s_{i_1})!} \dots \frac{1}{(n - s_{i_j})!} \\ &= \frac{1}{n!} \beta_n(S). \end{aligned}$$

$$\text{Thus } \beta_n(S) = \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq k} (-1)^{k-j} \binom{n}{s_{i_1}, s_{i_2} - s_{i_1}, \dots, n - s_{i_j}} = n! \det[1/(s_{j+1} - s_i)], (i, j) \in [0, k] \times [0, k].$$

Lets write down this matrix (row i, colon j) $\beta_n(S) =$

$$n! \begin{vmatrix} g(0,1) & g(0,2) & \dots & g(0,k+1) \\ g(1,1) & g(1,2) & \dots & g(1,k+1) \\ \dots & \dots & g(i,j+1) & \dots \\ g(k,1) & g(k,2) & \dots & g(k,k+1) \end{vmatrix} = n! \begin{vmatrix} \frac{1}{(s_1 - s_0)!} & \frac{1}{(s_2 - s_0)!} & \dots & \frac{1}{(s_{k+1} - s_0)!} \\ \frac{1}{(s_1 - s_1)!} & \frac{1}{(s_2 - s_1)!} & \dots & \frac{1}{(s_{k+1} - s_1)!} \\ \dots & \dots & \frac{1}{(s_{j+1} - s_i)!} & \dots \\ \frac{1}{(s_1 - s_k)!} & \frac{1}{(s_2 - s_k)!} & \dots & \frac{1}{(s_{k+1} - s_k)!} \end{vmatrix}$$

Let $e_1 e_2 \dots e_{n-1}$ be the binary string correspondning to a descent set T with $|T| = j$, thus T_j .

The Viennot triangle, the Boustrophedon transform and the descent statistic maximized

Now we shall study $\beta(S)$ by using the Viennot triangle. This will reveal more information of $\beta(S)$, for example for wich S, $\beta(S)$ is maximized.

Definition: Define the affine maps $\sum_x^{\rightarrow}(x)$ and $\sum_x^{\leftarrow}$ by

$$\sum_x^{\rightarrow}(x) = (x, x + x_1, x + x_1 + x_2, \dots, x + x_1 + \dots + x_n) \text{ and}$$

$$\sum_x^{\leftarrow}(x) = (x + x_1 + \dots + x_n, \dots, x + x_{n-1} + x_n, x + x_n, x)$$

Definition: Let $S \subseteq [n-1]$ and let $v_0, \dots, v_n$ be a sequence of vectors where $v_i \in \mathbb{R}^{i+1}$, such that $v_0 = (1)$ and

$$v_i = \begin{cases} \sum_x^{\leftarrow}(v_{i-1}) & \text{if } i \notin S \\ \sum_x^{\rightarrow}(v_{i-1}) & \text{if } i \in S \end{cases}$$

Proposition(Viennot): If we have the vector $v_{n-1} = (y_1, \dots, y_n)$ as the bottom row in the Viennot triangle then $\beta(S) = \sum_{k=1}^n y_k$.

Example: Lets verify the new formula by using the same descent sets as in prior example with $n - 1 = 8$ and $S = \{1, 5\}$.

Lets take a look at some values of $\beta(S)$.

Proposition: Denote the complementary set of S by $\bar{S}$ such that $\bar{S} = [n - 1] - S$. And the reverse set of S by S^* , such that $S^* = \{i : n - i \in S, 1 \leq i \leq n - 1\}$.

Then $\beta(S) = \beta(\bar{S}) = \beta(S^*)$.

Proof:

Definition: A run is a sequence of consecutive integers either all in S or all not in S

Example: With $S = \{1, 5\} \subseteq [7]$ We get the four runs $\{1\}, \{2, 3, 4\}, \{5\}, \{6, 7\}$. It is easy to see that S and $\bar{S}$ has the same runs wich follows directly from the definition. Therefore it is convenient to encode this in one way.

Definition: S and $\bar{S}$ encodes a list $L = (l_1, \dots, l_m), 1 \leq m \leq n - 1$ where l_m is the length of each run in the given order.

Example: Given the set $S = \{1, 5\}$ and its complement $\bar{S} = \{2, 3, 4, 6, 7\}$ we get the runs $\{1\}, \{2, 3, 4\}, \{5\}, \{6, 7\}$ so the corresponding list is $L = (1, 3, 1, 2)$

Given a list with repeated consecutive integers, foreample $L = (a_1, a_2, \dots, a_k)$ we write $L(a^k)$.

By taking a special case of the Viennot triangle we get the boustrophedon transform. This is the case where $S = \{1, 3, 5, \dots\} \cap [n - 1]$ or $S = \{2, 4, 6, \dots\} \cap [n - 1]$, in other words the descent statistics of the alternating permutations. We shall later, when we have established more results, show that this is the S that maximizes the descent set statistic $\beta(S)$.

Lets take a look at the triangular array within this triangle.

Definition: Let $E_{n,k} (n \geq k \geq 0)$ denote the number of permutations of $\{1, 2, \dots, n+1\}$, which alternarively fall and rise, that start with $k+1$ and a fall.

Proposition: $E_{n,k}$ satisfies the recursion

Definition: (The generalized boustrophedon transform). Let $S \subseteq \mathbb{Z}^+$. The sequence $b = (b_0, b_1, \dots)$ is called the boustrophedon transform of the sequence $a = (a_0, a_1, \dots)$, with respect to S , achieved by creating the following infinate array defined by the following recursion $\beta_{n,i}(S) = t_{n,i} \ 0 \leq i \leq n$: The initial condition $t_{n,0} = a_n$

$$\beta_{n,i} = \begin{cases} \beta_{n,i-1} + \beta_{n-1,i-1} & \text{if } n-1, n \in S \text{ or } n-1, n \notin S \\ \beta_{n,i-1} + \beta_{n-1,n-1} & \text{otherwise} \end{cases}$$

Set $b_n = \beta_{n,n}$.

Example: Give several examples of boustrophedon transformed sequences.

Definition: Let $S \subseteq \mathbb{Z}^+$ and $n, j \in \mathbb{N}$, define $S_{n,j} = \{s \in \{1, \dots, n - j\} : s + j \in S\}$

Definition: e_j denotes the j :th unit sequence $(0, 0, \dots, 0, 1, 0, \dots)$.

Proposition (main boustrophedon transform formula): Let $S \subseteq \mathbb{Z}^+$. The boustrophedon transform of the sequence $a = (a_0, a_1, \dots)$ with respect to S is a linear function given by

$$b_n = \sum_{j=0}^n \binom{n}{j} \beta(S_{n-1,j}) a_j.$$

When S is one of the alternating sequences $\{1, 3, 5, \dots\}$ or $\{2, 4, 6, \dots\}$ we get the special case with Euler numbers. Later on, when we have established more results, we will as mentioned before, show that this the case where S maximizes $\beta(S)$.

Lemma: Let $S \subseteq \mathbb{Z}^+$, where S is either $\{1, 3, 5, \dots\}$ or $\{2, 4, 6, \dots\}$. The boustrophedon transform of the sequence $a = (a_0, a_1, \dots)$ with S is given by $b_n = \sum_{j=0}^n \binom{n}{j} E_n(S_{n-1,j}) a_j$,

where E_n is the n :th Euler number.

We may express this as a generating function $\sum_{n \geq 0} b_n \frac{x^n}{n!} = (\tan(x) + \sec(x)) \cdot \sum_{n \geq 0} a_n \frac{x^n}{n!}$.

In order to proof the proposition and the lemma we need to establish some more results.

Definition: Let $S = \{s_1, \dots, s_k\} \subseteq [n]$ and let $\beta_i(S)$ be defined in two ways:

- 1) If $n \notin S$, $\beta_i(S) = \#\{\pi \in S_{n+1} : D(\pi) = S : s_k = i\}$
- 2) If $n \in S$, $\beta_i(S) = \#\{\pi \in S_{n+1} : D(\pi) = S : s_k = n - i + 2\}$

$\beta_i(S)$ are a refinement of $\beta(S)$ (explain partially ordered sets). We have $\beta(S) = \sum_{i=1}^{n+1} \beta_i(S)$ and

$$\beta(S \cap \{1, \dots, n-1\}) = \beta_{n+1}(S).$$

Lemma: Given the boustrophedon transform of the sequence $e_0 = (1, 0, 0, \dots)$ with regard to $S = \{s_1, \dots, s_k\} \subseteq [n]$

$$\Rightarrow t_{n,i}(S) = \begin{cases} \#\{\pi \in S_{n+1} : D(\pi) = S \cap [n] : s_k = i + 1\} & \text{if } n \notin S \\ \#\{\pi \in S_{n+1} : D(\pi) = S \cap [n] : s_k = n - i + 1\} & \text{if } n \in S \end{cases}$$

$$t_{n,n}(S) = \beta(S \cap [n-1])$$

In order to prove this lemma and proposition ($b_n = \sum_{j=0}^n \binom{n}{j} \beta(S_{n-1,j}) a_j$) we will first proof a

generalized version of the lemma above. But first we need some new definitions.

Definition: A partial permutation of length k on a set $[n]$ is an array of length k $\mathcal{Y} = \mathcal{Y}_1 \mathcal{Y}_2 \cdots \mathcal{Y}_k$ where the entries $\mathcal{Y}_i$ are from $[n]$ appear at most once.

Definition: Denote the set of all partial permutations of length k $\mathcal{Y} = \mathcal{Y}_1 \mathcal{Y}_2 \cdots \mathcal{Y}_k$ with entries $\mathcal{Y}_i$ from $[n]$ by P_n^k .

Definition: The descent set on a partial permutation $\mathcal{Y} = \mathcal{Y}_1 \mathcal{Y}_2 \cdots \mathcal{Y}_k$ on $[n]$ is denoted

$$D_p(\mathcal{Y}) = \{i : \mathcal{Y}_i > \mathcal{Y}_{i+1}\} \subseteq [k-1]$$

Proposition: The number of partial permutations $\mathcal{Y} = \mathcal{Y}_1 \mathcal{Y}_2 \cdots \mathcal{Y}_k$ on $[n]$ with descent set

$$D(\mathcal{Y}) = S \subseteq [k-1] \text{ is } \binom{n}{k} \beta(S).$$

Proof: (obvious)

Lemma (same as before but for j instead of 0)

Proof of $b_n = \sum_{j=0}^n \binom{n}{j} \beta(S_{n-1,j}) a_j$

Lemma: Given the boustrophedon transform of the sequence $e_j = (0, 0, \dots, 0, 1, 0, \dots)$ with j th position 1, regard to $S = \{s_1, \dots, s_k\} \subseteq [n]$

$$\Rightarrow \beta_{n,i}(S) = \begin{cases} \#\{\gamma \in S_{n+1}^{n-j+1} : D_p(\gamma) = S_{n,j} : s_k = i+1\} & \text{if } n \notin S \\ \#\{\gamma \in S_{n+1}^{n-j+1} : D_p(\gamma) = S_{n,j} : s_k = n-i+1\} & \text{if } n \in S \end{cases}$$

Proof: Let $i = 0$ then we have $n = j \Rightarrow \beta_{n,0} = 1$ and $n \neq j \Rightarrow \beta_{n,0} = 0$.

To establish $\beta_{n,i}$ we have to consider four cases:

- 1) $n-1, n \notin S$
- 2) $n-1 \notin S, n \in S$
- 3) $n-1 \in S, n \notin S$
- 4) $n-1, n \in S$

I will only do 1) because the other cases follow the same line of thoughts.

Let $P_{n,i}^{n-j+1} = \{\lambda \in P_n^{n-j+1} : D(\lambda) = S_{n,j}, \lambda_{n-j+1} = i+1\}$. We have $\beta_{n,i} = \#P_{n,i}^{n-j+1}$.

$n-1, n \notin S \Rightarrow n-j-1, n-j \notin S_{n,j} \Rightarrow \lambda_{n-j-1} < \lambda_{n-j} < \lambda_{n-j+1} = i+1$. So $P_{n,i}^{n-j+1}$ can be partitioned into the following disjoint sets:

$A = \{\lambda \in P_{n,i}^{n-j+1} : \lambda_{n-j} < i\}$ and

$A = \{\lambda \in P_{n,i}^{n-j+1} : \lambda_{n-j} = i\}$

$\phi_A : A \rightarrow P_{n,i-1}^{n-j+1}$ where $\phi_A(i+1) = i, \phi_A(i) = i+1$ and $\phi_A(\lambda_k) = \lambda_k$ otherwise. Thus ϕ_A is a bijection between A and $P_{n,i-1}^{n-j+1}$.

$\phi_B : B \rightarrow P_{n-1,i-1}^{n-j+1}$ where we remove $i+1$ and set $\phi_B(l) = l-1, i+2 \leq l \leq n+1$. Thus ϕ_B is a bijection between B and $P_{n-1,i-1}^{n-j+1}$. If we go through the other three cases for $n-1$ and n we establish the boustrophedon recursion:

$$\beta_{n,i} = \begin{cases} \beta_{n,i-1} + \beta_{n-1,i-1} & \text{if } n-1, n \in S \text{ or } n-1, n \notin S \\ \beta_{n,i-1} + \beta_{n-1,n-i} & \text{otherwise} \end{cases}$$

as desired.

Corollary: $\beta_{n,n}(S) = \binom{n}{j} \cdot \beta(S_{n-1,j})$

Proof: Set $i = n$. Then, by Lemma, $\beta_{n,n}(S) = \#\{\lambda \in P_n^{n-j} : D(\lambda) = S_{n-1,j}\}$.

Now we have what it takes to proof our main proposition of the boustrophedon e_j transform formula. Given $S \subseteq \mathbb{Z}^+$, the boustrophedon transform of the sequence $a = (a_0, a_1, \dots)$ with

respect to S is $b_n = \sum_{j=0}^n \binom{n}{j} \beta(S_{n-1,j}) a_j$.

Proof: The boustrophedon transform is a linear function so it only need to proof it for the sequence e_j . The proof for e_j follows directly from corollary.

Music and partially ordered sets

Partially ordered sets the basics

Definition: A *partially ordered set* P is a set together with a binary relation denoted $\leq_p$ satisfying:

- 1) $x \leq_p x, \forall x \in P$.(reflexivity)
- 2) $(x \leq_p y) \wedge (y \leq_p x) \Rightarrow x = y$.(antisymmetri)
- 3) $(x \leq_p y) \wedge (y \leq_p z) \Rightarrow x \leq_p z$.(transitivity)

If $x \leq_p y$ or $y \leq_p x$, we say that x and y are *comparable*, otherwise x and y are *imcomparable*.

Definition: A *weak subposet* Q of P is a subset Q of the elements of P and a partial ordering Such that $x \leq_p y$ in $Q \Rightarrow x \leq_p y$ in P .

Definition: if Q is a weak subposet of P with $P = Q$, then P is a refinement of Q .

Definition: induced subposet

Definition: The posets P and Q are *isomorphic* if there exists an order-invariant(define this) bijection $\phi: P \rightarrow Q$, where ϕ^{-1} is also order-invariant: $x \leq_p y$ in $P \Leftrightarrow \phi(x) \leq \phi(y)$ in Q .

Definition: An *interval* is a subposet $[x, y] = \{z \in P : x \leq_p z \leq_p y\}$ defined when $x, y \in P$. A poset where every interval is finite is called a *locally finite poset*.

Definition: If for $x, y \in P$, we have that $x <_p y$ and there is no z such that $x < z < y$, then y covers x

Definition: A Hassediagram is a graph of a finite poset P where the vertices are the elements of P and the edges are the cover relations such that if $x \leq_p y$ then y is placed "above" x

Duality

In the previous definitions, we often noted that a concept can be defined by just inverting the ordering in a former definition. This is the case for "least" and "greatest", for "minimal" and "maximal", for "upper bound" and "lower bound", and so on. This is a general situation in order theory: A given order can be inverted by just exchanging its direction, pictorially flipping the Hasse diagram top-down. This yields the so-called **dual, inverse, or opposite order**. Every order-theoretic definition has its dual: it is the notion one obtains by applying the definition to the inverse order. Since the symmetry of all concepts, this operation preserves the theorems of partial orders. For a given mathematical result, one can just invert the order and replace all definitions by their duals and one obtains another valid theorem. This is important and useful, since one obtains two theorems for the price of one.

Definition: A chain is a poset where any two elements are comparable. Thus a *totally ordered poset*.

Definition: The rank $l(C)$ of a finite chain is defined as $l(C) = |C| - 1$.

Definition: The rank of a finite poset is $l(P) := \max\{l(C) : C \text{ is a chain in } P\}$

Definition An *antichain* is a subset A of a poset P where all elements are incomparable.

Definition: An order ideal I of a poset P is a subset I of P where $x \in I \wedge y \geq_p x \Rightarrow y \in I$.

When $\#P < \infty$ there is a one-to-one correspondence between the antichains A and the ideals I in P . An antichain A is the set of maximal elements of an order ideal I and conversely

$I = \{x \in P : x \leq_p y, y \in A\}$ (Forexample if there are two maximal elements in I they can ofcourse not be compareable since then only one of them would be maximal, thus they must be imcomparable).

Definition: If $A = \{y_1, \dots, y_k\}$ and $I = \{x \in P : x \leq_p y_i, y_i \in A\}$ then we say that I is *generated by* A and denote this by $\langle y_1, \dots, y_k \rangle = I$.

Definition: The set of all order Ideals ordered by inclusion is a poset denoted $J(P)$.

(Some operations on partially ordered sets)

Definition: Let P and Q be posets where $P \cap Q = \emptyset$ as sets. The *direct sum* of P and Q is the poset $P + Q$ on the union $P \cup Q$, where $x \leq_p y$ in $P + Q$ if either $x \leq_p y$ in P or $x \leq_p y$ in Q .

Example The disjoint union of P of P with itself n times is denoted nP . If $\mathbf{n}$ is an n -element antichain then we have $\mathbf{n} \cong n\mathbf{1}$.

Definition: Let P and Q be posets, disjoint as sets. Then the *ordinal sum* of P and Q is the poset $P \oplus Q$ on the union $P \cup Q$ such that $x \leq_p y$ in $P \oplus Q$ if either

- 1) $x \leq_p y$ in $P, x, y \in P$
- 2) $x \leq_p y$ in $Q, x, y \in Q$
- 3) $x \in P$ and $y \in Q$

Forexample an n -element chain is $\mathbf{n} = \mathbf{1} \oplus \mathbf{1} \oplus \dots \oplus \mathbf{1}$

Definition: The *direct product* of P and Q is the poset $P \times Q$ on the set of tuples (x, y) with $x \in Q$ and $y \in P$ such that $x \leq x'$ in P and $y \leq_p y' \Rightarrow (x, y) \leq_p (x', y')$ in $P \times Q$.

We will now introduce an important class of posets and look into the operations it is based on. Given $x, y \in P$, the *upper bound* of x and y is an element z satisfying $x \leq_p z$ and $y \leq_p z$.

The *least upper bound* is an element $w = \min\{z \in P : x \leq_p z, y \leq_p z\}$. We denote this element $x \vee y$, read x join y . Dually we can define the *greatest lower bound*, denoted $x \wedge y$, read x meet y , by $x \wedge y = \max\{z \in P : z \leq_p x, z \leq_p y\}$

Definition: A *lattice* L is a poset where every two elements has a least upper bound and a greatest lower bound. In algebraic terms, L is a set together with two binary operations $\wedge$ and $\vee$ that are associative, commutative, idempotent and satisfies the absorption law.

Definition: A *distributive lattice* is a lattice L with satisfies the distributive laws. We have $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \Leftrightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x, y, z \in L$.

Definition: An element x of a lattice L is *join-irreducible* if we cannot write $x = y \vee z$ such that $y \leq_p x$ and $z \leq_p x$. An order ideal I in P is join-irreducible in $J(P) \Leftrightarrow I$ is a principal order ideal in P . Denote a join-irreducible ideal by $\wedge_x$.

Proposition: The induced subposets Q of join-irreducible ideals of $J(P)$ is isomorphic to P . Hence $J(P) \cong J(Q) \Leftrightarrow P \cong Q$.

Proof: Since an order ideal I in P is join-irreducible in $J(P) \Leftrightarrow I$ is a principal order ideal in P , there is a one-to-one correspondence between the join-irreducibles $\wedge_x$ of $J(P)$ and the elements x of P . $\wedge_x \subseteq \wedge_y \Leftrightarrow x \leq_p y$. $\square$

Fundamental theorem for finite distributive lattices (FTFDL)

Let L be a finite distributive lattice. Then there is a unique, up to isomorphism, finite poset such that $L \cong J(P)$.

Proof: Because of proposition we only need to show that if P is the subposet of join-irreducibles of L , then $L \cong J(P)$. Define a mapping $\phi: L \rightarrow J(P)$ by $\phi(x) = I_x$.

We need to show this is a order-preserving bijection. We have $\phi(x) = I_x$ and $\phi(y) = I_y$ so $x \leq_p y$ in L gives us $I_x \leq_p I_y$ in $J(P)$. $\phi(x) = \phi(y) \Leftrightarrow I_x = I_y \Rightarrow x = y$, so ϕ is injective.

We will now show that ϕ is surjective: Given $I \in J(P)$ and $x = \bigvee \{y : y \in I\}$ we shall show that $I = I_x$. We will do this by showing $I \subseteq I_x$ and $I_x \subseteq I$. It's easy to see that $I \subseteq I_x$ so $\bigvee \{y : y \in I\} = \bigvee \{y : y \in I_x\}$. Let $z \in I_x$. Since we are working with a distributive lattice we have (b) $\bigvee \{y \wedge z : y \in I\} = \bigvee \{y \wedge z : y \in I_x\}$. $\bigvee \{y \wedge z : y \in I\} = z$ since all other elements are $\leq z$. Since we are working with a join-irreducible subposet the element z is join-irreducible and so by (b) we get that for a $y \in I$ with $z \leq_p y$ we have $y \wedge z = z$. I is an order Ideal, so $z \in I$ and thus $I_x \subseteq I$. So we have established that $I = I_x$. $\square$

Proposition: Let P be a n -element poset, then $J(P)$ is graded of rank n and $\rho(I) = \#I, I \in J(P)$, when I is regarded as an order ideal of P .

Proof: Regard I as an order ideal of P . Then the elements of $J(P)$ that cover I are the order ideals $I \cup \{x\}$, where x is $\min\{x \in P - I\}$.

By proposition and proposition $\Rightarrow P \sim J(P)$ so

$\#\{\text{non-isomorphic posets } P : \#P = n\} = \#\{\text{non-isomorphic lattices } L : \rho(L) = n\}$.

If we take a look at the n -element antichain $P = n \mathbf{1}$, any subset of P is an order ideal. $J(P)$ is the set of subsets of P , ordered by inclusion. We write $B_n = J(n \mathbf{1})$ also known as the *boolean algebra of rank n* .

Chains in distributive Lattices p 107 end of p 110 important

Many interesting enumerative properties have interpretations in terms of $J(P)$. We shall examine one that will be useful to us.

Let $\mathbf{m}^P$ denote the set of surjective order-preserving maps $\sigma : P \rightarrow \mathbf{m}$ and let $\mathbf{M}$ denote the set of chains $\hat{0} = I_0 < I_1 < \dots < I_m = \hat{1}$ of length m in $J(P)$.

Proposition: Let $m \in \mathbb{N}$ and let P be a finite poset. Then $\#\mathbf{m}^P = \#\mathbf{M}$.

Proof

Definition: Let $\#P = n$. Then an order-preserving bijection $\sigma : P \rightarrow \mathbf{n}$ is called an *extension of P to a total order*. Let $\mathbf{n}^P$ denote the set of all these extensions of P by $\mathbf{n}^P$ and denote $\#\mathbf{n}^P$ by $e(P)$. **p111**[[The number of lattice paths? (p111) we may identify each extension with a permutation $\sigma^{-1}(1), \dots, \sigma(n)$]]

Let P be the partial ordering on the set $[n]$ and consider the finite distributive lattice $L = J(P)$. By proposition we have an extension $\sigma : P \rightarrow [n]$ of P and we may identify each extension with a permutation $\sigma^{-1}(1), \dots, \sigma(n)$ of $[n]$.

Definition: The set of all permutations obtained as above is called the *Jordan-Hölder set* of P . Let's denote this $L(P)$

Example: (give fig Hassediagram etc)

Definition: Let P be a locally finite poset, Let K be a field and let $\text{Int}(P)$ denote the set of intervals of P . The *incidence algebra* $I(P, K)$ of P over K is the K -Algebra of all functions

$f : \text{Int}(P) \rightarrow K$. Multiplication here is defined by $fg([x, y]) = \sum_{x \leq z \leq y} f([x, z])g([z, y])$.

$I(P, K)$ is associative and has a two-sided identity defined by

$$\delta([x, y]) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

We will consider cases where $K = \mathbb{C}$ and therefore denote $I(P, \mathbb{C})$ simply as $I(P)$.

Proposition(p 114 Stanley) Let $f \in I(P)$. Then the following statements are equivalent

- 1)f has a left inverse
- 2)f has a right inverse
- 3)f has a two sided inverse
- 4) $f(x, x) \neq 0, \forall x \in P$

If f^{-1} exists then $f^{-1}(x, y)$ only depends on $[x, y]$

Proof:

Definition: The Zeta function ζ is defined as follows

$\zeta : \text{Int}(P) \rightarrow K$ by $\zeta([x, y]) = 1$, if $x \leq y$ in P .

By proposition we get that the zeta function ζ of a locally finite poset P is invertible.

Definition: Let the Möbius function of P be denoted by μ_P . We have

$$\mu_P \zeta = \delta \Leftrightarrow \mu(x, x) = 1, \forall x \in P \text{ and } \mu_P(x, y) = - \sum_{x \leq z < y} \mu_P(x, z), \forall x < y \text{ in } P$$

(Mera exempel och förklaringar inför vad inversen av Zeta funktionen betyder, nämn även koppling till topologi)

Proposition: (Möbius inversion formula) Let P be a poset. Let every principal order ideal in P be finite and let $f, g : P \rightarrow \mathbb{C}$. Then

$$f(x) = \sum_{y \leq x} f(y), \forall x \in P \Leftrightarrow f(x) = \sum_{y \leq x} g(y) \mu_P(y, x), \forall x \in P.$$

Proof (p116) $I(P)$ acts (on the right) as an algebra of linear transformations on the set $\mathbb{C}^P$ of all functions $P \rightarrow \mathbb{C} : (f\gamma)(x) = \sum_{y \leq x} f(y)\gamma([x, y])$, $f \in \mathbb{C}^P, \gamma \in I(P)$. Thus

$$f\zeta = g \Leftrightarrow f = g\mu_P. \quad \square$$

Proposition (Möbius inversion formula dual version) Let P be a poset. Let every dual principal order ideal in P be finite and let $f, g : P \rightarrow \mathbb{C}$. Then

$$g(x) = \sum_{y \geq x} f(y), \forall x \in P \Leftrightarrow f(x) = \sum_{y \geq x} \mu_P(y, x)g(y), \forall x \in P$$

Proof: The same line of thought as the above proof but $I(P)$ acts on the left. $\square$

Definition (if not earlier) Let $\hat{P}$ denote the poset consisting of the finite poset P with a $\hat{0}$ and a $\hat{1}$ adjoined.

Definition: Define c_i to be the number of chains $\hat{0} = x_0 < x_1 < \dots < x_i = \hat{1}$ with $\ell(\hat{0}, \hat{1}) = i$.

Proposition: $\mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - c_3 + \dots$, where $\hat{P}$ and c_i is as defined above.

Proof:

$$\begin{aligned} \mu_{\hat{P}}(\hat{0}, \hat{1}) &= (1 + (\zeta - 1)^{-1})(\hat{0}, \hat{1}) = (1 - (\zeta - 1) + (\zeta - 1)^2 - \dots)(\hat{0}, \hat{1}) = \delta(\hat{0}, \hat{1}) - (\zeta - 1)(\hat{0}, \hat{1}) + (\zeta - 1)^2(\hat{0}, \hat{1}) - \dots = \\ &= c_0 - c_1 + c_2 - c_3 + \dots \end{aligned}$$

$\square$

Proposition: (the product theorem) Let P and Q are locally finite posets with direct product $P \times Q$. $(x, y) \leq_P (x', y')$ in $P \times Q \Rightarrow \mu_{P \times Q}((x, y), (x', y')) = \mu_P(x, x')\mu_Q(y, y')$.

Proof: $\sum_{(x, y) \leq (u, v) \leq (x', y')} \mu_P(x, u)\mu_Q(y, v) = \sum_{x \leq u \leq x'} \mu_P(x, u) \sum_{y \leq v \leq y'} \mu_Q(y, v) = \delta_{xx'}\delta_{yy'} = \delta_{(x, y)(x', y')}.$

Proof: As the above but $I(P)$ acts on the left: $(\gamma f)(x) = \sum_{y \geq x} \gamma([x, y]) f(y)$.

Let $P = B_n$. $B_n \cong 2^n$. The Möbius function on 2 is $\mu(1, 1) = \mu(2, 2) = 1, \mu(1, 2) = -1$. Think of B_n as the set of all subsets of an n -set X and consider the length $\ell(T, S) = |S - T| \Rightarrow$

$\mu(T, S) = (-1)^{\ell(T, S)}$. We can present a Möbius formula for B_n :

$g(S) = \sum_{T \subseteq S} f(T), S \subseteq X \Leftrightarrow f(S) = \sum_{T \subseteq S} (-1)^{\ell(T, S)} g(T), S \subseteq X$. Which is infact equivalent to the

principle of inclusion exclusion formula $f_{\leq}(T) = \sum_{Y \subseteq T} (-1)^{|T-Y|} f_{\leq}(Y)$. We conclude that the

Möbius inversion formula on a boolean algebra is equivalent to the principle of Inclusion-Exclusion. And as you might have suspected we arrive att computing the descent statistic with the Möbius inversin formula.

(Proposition: p 119)

Rank-selection Möbius inversion formula and descent statistic

Definition: A S -rank-selected subposet of P is a subposet $P_S = \{x \in P : \rho(x) \in S\}$, where $S \subseteq [0, n]$ and P is a finite graded poset of rank n , with rankfunction $\rho : P \rightarrow [0, n]$.

Now lets redefine the inclusive descent set statistic $\alpha(S)$

Definition: $\alpha(P, S) = \alpha(S) :=$ the number of maximal chains of P_S . Forexample $\alpha(\{i\}) =$ the number of elements in P with rank i .

And so by redefinig $\beta(P, S) = \beta(S)$.

Definition: $\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha(T) \quad \Leftrightarrow \quad \alpha(S) = \sum_{T \subseteq S} \beta(T)$
the Principle of Inclusion-Exclusion

Consider the Möbius function μ_S on the poset $\hat{P}$. Then by proposition we get

$$\beta(S) = (-1)^{|S|-1} \mu_S(\hat{0}, \hat{1}).$$

Theorem: $L = J(P)$ and $S \subseteq [n-1] \Rightarrow \beta(L, S) = \#\{\pi \in L(P) : D(\pi) = S\}$

Proof: (p 132)

Corollary: Let $L = B_n$ and $S \subseteq [n-1] \Rightarrow \beta(L, S) = \beta_n(S)$. $\square$

Parikh matriser

This introduces an interesting connection between the theory of music and the theory of formal languages. I will not go deeply into syntax or grammar of music but present a few tools of analysing melodies and there structure using various Parikh mapping such as parix matrices. Expanding this mapping will give more information about the structure of melodies. q-expansion can be interpreted as “musical timing”.

When we analyse music we may diregard the natural order of time and just study the natural order of pitches for example. Then we may go back and forth in both two directions as we go in one direction through the natural order of pitches.

Cause a effect – causality – induction – if we move melody up we will get higher in pitch

If move melody down we will get lower in pitch. Cause – move up or down – effect get

higher or lower. Conclusion is last effect and hence causes no effect. The start is the first

effect and has no cause. There is time and there is order of cause and effect. They may not

coincide. In music we might view the highest note as the musical peak with regard to level of

pitch. Timing is the concept that considers when this occurs in relation to the other events

(each note occuring being an event) . If we consider the *timing* of three musical notes

in relation to 9 others, all 12 distinct. They all might appear together in the middle or spread out with maximal distans in the melody. We may define balance musical equilibrium of timing as the 0 difference between two sets like this in timing. That is the sum of index representing the timing of a set while the. So if the absolute belopp of the index sum of one set of pitches – an other equals the balance of the it is sort of a distance function of a metric space. Anyways, the the entries of the parikh q-matrix is sum of indexes of the letters of the scattered subwords of the word in question.

When compising or analysing music by it's non sounding representation we have the advantage of note being in coincidence with the natural order in time. We may invent things in the music that is above or independent of the natural order of time and has an causality order that is different than the natural order than time. If forexample 1234 represent a melody that corresponds with time then 1423 is one independent from the natural order of time. (this is an abstack idea that is very interesting and changes the way of thinking and creating music).

Some background. The set of *words* over a given *alphabet* is a *semigroup* under the operation of *concatenation* of words. Let A and B be semigroups. We have the morphism: $\phi: A \rightarrow B$, $\phi(uv) = \phi(u)\phi(v)$. If a semigroup A as an element ε such that $\varepsilon u = u\varepsilon = u$, $\forall u \in A$.

Definition: A semigroup is a set together with a binary operation that is associative.

Definition: A *monoid* is a semigroup with an element ε such that $\varepsilon u = u\varepsilon = u$, $\forall u \in A$.

Definition: Let us define an alphabet to be a totally ordered finite set denoted by $\Sigma = \{a_1 < \dots < a_k\}$.

Definition: a word w is a permutation of the letters in Σ . Lets denote the set of all words with letters in Σ by Σ^* and denote the set of all words of length n with letters in Σ by Σ_n^* .

In this paper we will study a certain cases of words. Lets clarify our definitions for these words:

Definition: Let the unordered set of letters in $\Sigma = \{a_1 < \dots < a_k\}$ be denoted by $A = \{a_1, \dots, a_k\}$. Define a function $\varphi: A \rightarrow N$ by $\varphi(a_i) = i$. Clearly φ is a bijection. Our alphabet is transformed into $\varphi(\Sigma) = \{\varphi(a_1), \dots, \varphi(a_k)\}_< = \{1, \dots, k\}_<$, thus φ is order preserving and defines a poset isomorphism from A to N.

Definition: Let $u = x_1 \cdots x_k$ be a word such that x_i are words, some of them possibly empty. Then the words x_i are *factor subwords* of u.

Definition: Let $u = u_1 \cdots u_r$ and $v = v_0 \cdots v_r$ be words such that v_i and u_i are factor subwords, possibly empty. Then in the word $w = u_0 v_1 u_1 v_2 \cdots u_{r-1} v_{r-1} u_r v_r$ we have that u and v are *scattered subwords* of w.

Definition: Let the length of the word be defined as the number of letters of the word denoted by $|w| = |x_1 \cdots x_n| = n$ where $x_i \in \Sigma$.

Definition: The number of occurrences of a word u as a scattered subword of w is denoted by $|w|_u$. If u is the empty word $u = \lambda$ then by convention we have $|u|_\lambda = 1$. If $|w| < |u|$ then $|w|_u = 0$.

Example:

Definition: Let M denote a multiset of $N = \{1, \dots, k\}$ such that $\phi(i) = b_i$ number of occurrences of element i . We have $M = \{1^{b_1}, \dots, k^{b_k}\}$. If we take an arbitrary word $w \in \Sigma^*$ it may or may not contain else than exactly one occurrence all the letters of Σ . Lets denote the number of occurrences of letter a_i by $|w|_{a_i}$.

Definition: The *Parikh vector mapping* is a morphism $\psi_1 : \Sigma^* \rightarrow \mathbb{N}^k$ defined by $\psi_1(w) = (|w|_{a_1}, \dots, |w|_{a_k})$. Given $w_1, w_2 \in \Sigma^*$ we have $\psi_1(w_1 * w_2) = \psi_1(w_1) + \psi_1(w_2)$, where $*$ denotes concatenation.

Consider the alphabet defined above $\{1, \dots, k\}_<$. Given any word $w \in \Sigma_n^*$ there is a bijection $\theta : \mathbb{N}^k \rightarrow \forall M_N$, where $\forall M$ denotes the set of all multisets of N , defined by

$\theta(\psi_1(w)) = \theta(|w|_{a_1}, \dots, |w|_{a_k}) = \{1^{b_1}, \dots, k^{b_k}\}$ such that $\# \theta(\psi_1(w)) = \# M = n$. We have that

$\Sigma_n^* = S_n(M)$ and hence $\# \Sigma_n^* = \# S_n(M) = \binom{n}{b_1, \dots, b_k}$ and as we have a generating function for

the number of multisets M of N $\sum_{M \text{ of } N} x^{|M|} = \sum_{k \geq 0} \left(\binom{n}{k} \right) x^k = \sum_{k \geq 0} \binom{n+k-1}{k}$ and so by our

bijection θ we see that the number of parikh mappings as defined above is given by this generating function $\sum_{M \text{ of } N} x^{|M|}$.

Definition: Consider the alphabet $\Sigma = \{a_1 < \dots < a_k\}$. Then the $a_{i,j}$ denotes the word $a_{i,j} = a_i a_{i+1} \dots a_j$ of consecutive letters from the alphabet.

If we're looking at our transformed alphabet $\varphi(\Sigma) = \{1, \dots, k\}_<$ then we have

$\varphi(a_{i,j}) = i, j = i(i+1) \dots j$.

Definition: The number of occurrences of $a_{i,j}$ as a scattered subword of w is denoted by $|w|_{a_{i,j}}$.

Definition: Let $\Sigma = \{a_1, \dots, a_k\}_<$ be the an ordered alphabet.

The *parikh matrix mapping* is a morphism $\psi_{M_k} : \Sigma^* \rightarrow M_{k+1}$, where M_{k+1} denotes the collection of upper triangular $(k+1) \times (k+1)$ matrices $(m_{i,j})_{1 \leq i, j \leq k+1}$, such that $m_{i,j} \in \mathbb{N}$, defined by $\psi_{M_k}(a_l) = (m_{i,j})_{1 \leq i, j \leq k+1}$, then for each $1 \leq i \leq k+1$ we have $m_{i,i} = 1$, $m_{l,l+1} = 1$ and for all other entries in $\psi_{M_k}(a_l)$ we have $m_{i,j} = 0$. M_{k+1} is a monoid with respect to multiplication of matrices and has a unit wich is the identity matrix I_k .

We have $\psi_{M_k}(a_r a_t) = \psi_{M_k}(a_r) \psi_{M_k}(a_t)$, $a_r, a_t \in \Sigma$.

In other words we define parikh matrix mapping on letters and then by working in a monoid we can expand the mapping by where concatenation morphes to multiplication of matrices.

Theorem: Let $\Sigma = \{a_1, \dots, a_k\}_<$ be the an ordered alphabet with $k \geq 1$ and $w \in \Sigma^*$ then the matrix $\psi_{M_k}(w) = (m_{i,j})_{1 \leq i, j \leq k+1}$, has the following properties

- 1) $m_{i,j} = 0, \forall 1 \leq j < i \leq k+1$,
- 2) $m_{i,i} = 1, \forall 1 \leq i \leq k+1$
- 3) $m_{i,j+1} = |w|_{a_{i,j}}, \forall 1 \leq i \leq j \leq k+1$.

Definition: *j:th diagonal*

Corollary: Let $\Sigma = \{a_1, \dots, a_k\}_<$ be the an ordered alphabet. The Parikh matrix $\psi_{M_k}(w)$ has as its *second diagonal* the parikh vector of w . That is $(m_{1,2}, m_{2,3}, \dots, m_{k,k+1}) = \psi_1(|w|_{a_1}, \dots, |w|_{a_k})$

Definition (generalization of the parikh vector mapping) Let $\Sigma = \{a_1, \dots, a_k\}_<$ be the an ordered alphabet. Define the *parikh vector mapping of order j* as the mapping

$\psi_j : \Sigma^* \rightarrow \mathbb{N}^{k-j+1}$ by $\psi_j(w) = (|w|_{a_{1,j}}, |w|_{a_{2,j+1}}, \dots, |w|_{a_{k-j+1,j+k-1}})$. Where $j=1$ gives the “standard”

parikh vector mapping and $j=0$ is defined to be $\psi_0 = (|w|_{a_{1,0}}, |w|_{a_{2,1}}, \dots, |w|_{a_{k+1,k}}) = (\frac{1}{k+1}, \dots, 1)$.

(Although $|w|_{a_{i,j}}, i > j$ is not in general defined as 1 in any other case).

Corollary (of the generalized parikh vector mapping): Let $\Sigma = \{a_1, \dots, a_k\}_<$ be the an ordered alphabet. The Parikh matrix $\psi_{M_k}(w)$ has as its *j:th diagonal*, the *j:th order* parikh vector of w .

That is $(m_{1,j+1}, m_{2,j+2}, \dots, m_{k-j+1,j+k-1}) = \psi_j(w) = (|w|_{a_{1,j}}, |w|_{a_{2,j+1}}, \dots, |w|_{a_{k-j+1,j+k-1}})$.

(investigate whether or not this is a morphism. Closed under concatenation? Answer – no. But rather a recursive relation on the vectors of lower order)

investigate the Parikh Determinant to get the result that every det is nonnegative, thus a inverse *parikh-“meta” matrix* (“*imaginary parikh*” matrix) is defined (although it is not a member of parikh matrices since no word is mapped to a parikh matrix with negative entries M_k . So in a way this element of M_k . Reminds of the imaginary numbers as solutions to certain equations with coefficients that are reals.

Lemma: From determinant we can get an inequality that shows a recursive relationship between entries of third diagonal and second diagonal.

Definitions (theirs in relation to mine): *Parikh matrix equivalent words*, (matrix ambiguity) *ambiguous/unambiguous parikh matrices* – parikh set / parikh matrix set. Counting parikh equivalent words. Conclude with the Fibonnaci sequence parikh matrix and the pascal triangle parikh matrix give the musical notation of the words. (J.S Bach example, motivation for studying Bach’s music with *parikh matrix analysis* of fuge themes page 14 in Mateescu and Salomaa’s paper)

Explain: first-, second-,etc diagonal of the parikh matrix. Or simply represent them as Parikh vectors of first-,second-, etc order. First order being the standard parikh vector.

Background: We will get more information about the melodies from the parikh matrix.

We will make use of the theory of context free langague. Lets treat music as a context-free language. We will now consider melodies as words of an ordered alphabet (in our case mostly $\{0 < 1 < 2 < \dots < 12\}$ but not necessarily). All matrices are such that the left to right diagonal is defined as containing 1’s and the entries underneath it are 0’s. The enumeration is sometimes referred to as *subword histories of subwords*

Define subword histories of subwords

The meaning of the Matrix is constructed as a Theorem. The Parikh mapping (vector) is not injective, neither is the parikh matrix mapping or *Parikh q- Matrix*, however they are all morphism. Having a injective mapping mean that the information given by the parikh matrix would be “complete”.

But by extending the Parikh matrix mapping to a so-called *Parikh q-encoding matrix*, we have an injective mapp and thus can construct an inverse matrix however it is not a morphism. So the choice af “Parikh-tool” to study melodies depends on what we are interested in.

Parikh theorem: The image of the parikh mapping of a context free language is always a semi-linear set.

Basic idea is to give numerical properies of words expressed as a vector.

Number of occurences of letters, order of letters in a word, Number of occureses of *scattered subwords* of consecutively ascending order in a word.

Define scattered subwords. And subword (not sacttered if necessary)

Basic properties of parikh matrices:

Commutative omitt showing.

Parikh matrix theore m

Corollary1 Parikh vectors as second diagonal

Corollary2. generalization to parikh vectors of order j, where corollary1 is a special case.

Inverse Parikh matrix

The set of upper triangular matrices of parikh matrix typ is a noncommutative group with respect to multiplication. So every parikh matrix has an inverse. Although the Parikh matrix mapping does not necessarily have one.

Define the *alternate Parikh matrix* of a Parikh matrix A.

1) the Parikh Theorem is a conjecture about parkh vectors howrever it does NOT hold for Parikh matrix mapping

$$\text{ex } \psi(a^n b^n) = \begin{pmatrix} 1 & n & n^2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$$

Definition: Mirror word

(Theorem) Just a pre-comment. parikh vector of a *mirror word*

Theorem The inverse of a Parikh matrix of a word w and the alternate parikh matrix of the mirror image mi(w) are equal.

This gives us a beautifully simple way of computing the inverse of a Parikh matrix (music begrepp, if we surjective mappings of equivalence classes with cardinality >1 we may think of objects as substitutable by there equivalent objects. This gives us an automatisizable way of rearranging music by substituting their objects by their equivalents)

Mathematical music theory among many things provides composers with methods of performing operations/transformations of music such that some of its mathematical properties are unchanged but some are changed so in a sense the result is a new version equivalent to the original, that is only one type of use of mathematics to music.-)

Some furthure **corollaries**

In speculations define some sort of melodical distance function or metric space. By defining the distance between to words though parikh vectors or parikh matrices or their q-expansion. We have different levels, hiarcies of distance. Parikh space for melodies and how far away two melodies are from each other in this parikh space.

Then on to **q-expansion**

Begin by making q-expansion definition of the Parikh matrix (encoding as supposed to mapping) and the vector. We will show that the q-matrix mapping is injective.

Show how we q-count scattered subwords of words with respect to there positions.

Keeping track of positions of letters (in parikh matrix one does not distinguish to scattered subwords from each other, this is achived if we keep track of positions of the letters of the scattered subword) positions are regarded as exponents of monomials in q. when evaluated at $q = 1$ we get the original parikh matrix as a special case.

Definition and examples of Parikh q-Matrix encoding

The q-analog of the parkih matrix theorem...

Proof by induction

Injectivity of the q-matrix encoding

Give examples that show that injectivity is not satisfied for parikh matrix mappings and parikh q matrix mappings.

Theorem q- matrix **encoding** is injective. Mention that the q-analog of the theorem about matrix inverse and mirror image words holds. Give an example of a q-matrix encoding of an inverse...

If we consider the parikh mapping $\psi_{M_k} : S_n(w) \rightarrow M_{k+1}$, where $w \in S_n$ its obvious that the resulting parikh matrices have entries that are 1 or 0. Futhermore we get the following results:

Definition factor

Notation k,l

k,l exists in w. Means k,l exists as a scattered subword in w

Lemma: Let $\psi_{M_k} : S_k \rightarrow M_{k+1}$, defined by

$$\psi_{M_k}(\pi) = \begin{bmatrix} 1 & 1 & |\pi|_{1,2} & |\pi|_{1,2,3} & |\pi|_{1,2,3,4} & \dots & |\pi|_{1,\dots,k-1} & |\pi|_{1,\dots,k} \\ 0 & 1 & 1 & |\pi|_{2,3} & |\pi|_{2,3,4} & \dots & \dots & |\pi|_{2,\dots,k} \\ 0 & 0 & 1 & 1 & |\pi|_{3,4} & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 & \dots & \dots & |\pi|_{k-2,k-1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & |\pi|_{k-2,k-1} & |\pi|_{k-2,k-1,k} \\ \dots & \dots & \dots & \dots & \dots & 1 & 1 & |\pi|_{k-1,k} \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

where $\pi \in S_k$. Then $m_{i,j+1} = 1 \Leftrightarrow m_{i,j} = 1$ and $m_{i+1,j+1} = 1$, otherwise $m_{i,j+1} = 0$. $1 \leq i \leq k-1$, $3 \leq j \leq k+1$, $i+1 \leq j$.

Proof: Let $\pi \in S_k$.

1) Assume that $m_{i,j+1} = |\pi|_{i,j} = 1$ and $m_{i,j} = |\pi|_{i,j-1} = 0$ or $m_{i+1,j+1} = |\pi|_{i+1,j} = 0$ or both.

Then either $i, j-1$ or $i+1, j$ or both do not exist. Since $i, j-1$ and $i+1, j$ both are factors i, j this means that i, j cannot exist, so $m_{i,j+1} = |\pi|_{i,j} = 0$. But this contradicts our assumption

$m_{i,j+1} = |\pi|_{i,j} = 1$. Thus

$m_{i,j+1} = 1 \Rightarrow m_{i,j} = 1$ and $m_{i+1,j+1} = 1$

2) Assume that $m_{i,j} = |\pi|_{i,j-1} = 1$, $m_{i+1,j+1} = |\pi|_{i+1,j} = 1$ and $m_{i,j+1} = |\pi|_{i,j} = 0$. Then $i, j-1$ exists and $i+1, j$ exists and therefore i, j must exist, so $m_{i,j+1} = |\pi|_{i,j} = 1$. But this contradicts $m_{i,j+1} = 0$, so $m_{i,j+1} = 1 \Leftarrow m_{i,j} = 1$ and $m_{i+1,j+1} = 1$. $\square$

Corollary: Let $\psi_2 : S_k \rightarrow \mathbb{Z}_2^{k-1}$, $\psi_2(\pi) = (|\pi|_{1,2}, \dots, |\pi|_{k-1,k})$, $\pi \in S_k$. Then the Parikh matrix mapping $\psi_{M_k}(\pi)$ is completely determined by $\psi_2(\pi)$.

Proof: By the prior lemma we see that if an entry in the third diagonal or above (diagonally) is 0 then all entries to the left and above it will be 0. If we go through this procedure from the top diagonal entry $m_{1,3} = |\pi|_{1,2}$ to the bottom diagonal element $m_{k-1,k+1} = |\pi|_{k-1,k}$ and apply lemma as above, we see all entries are covered and thus the whole parikh matrix $\psi_{M_k}(\pi)$ is determined by $\psi_2(\pi)$. $\square$

A consequence of this result is that if we are able to show something for $\psi_2(\pi)$, it is directly expanded to $\psi_{M_k}(\pi)$. So if we are to show that $\psi_2(\pi)$ and $D(\pi)$ are in bijection this means that $D(\pi)$ and $\psi_{M_k}(\pi)$ are in bijection.

Computations

Parikh matrices:

Let $\Sigma_4 = \{1 < 2 < 3 < 4\}$ and $\Sigma^* = S_4$. The parikh mapping $\psi_{M_4} : S_4 \rightarrow M_5$ defined by

$$\psi_{M_5}(\pi) = \begin{bmatrix} 1 & |\pi|_1 & |\pi|_{12} & |\pi|_{123} & |\pi|_{1234} \\ 0 & 1 & |\pi|_2 & |\pi|_{23} & |\pi|_{234} \\ 0 & 0 & 1 & |\pi|_3 & |\pi|_{34} \\ 0 & 0 & 0 & 1 & |\pi|_4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\text{We have } \psi_{M_5}(1234) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \psi_{M_5}(1243) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\psi_{M_4}(1324) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \psi_{M_5}(1342) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\psi_{M_4}(1423) = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \psi_{M_4}(1432) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[illegible]

$$\begin{aligned}\psi_{M_4}(4213) &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \psi_{M_4}(4231) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \psi_{M_4}(4312) &= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \psi_{M_4}(4321) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

Parikh vectors of second order:

$$\psi_2(1234) = 111 \quad \psi_2(1243) = 110 \quad \psi_2(1324) = 101 \quad \psi_2(1342) = 101 \quad \psi_2(1423) = 110 \quad \psi_2(1432) = 100$$

$$\psi_2(2134) = 011 \quad \psi_2(2143) = 010 \quad \psi_2(2314) = 011 \quad \psi_2(2341) = 011 \quad \psi_2(2413) = 010$$

$$\psi_2(2431) = 010 \quad \psi_2(3124) = 101 \quad \psi_2(3142) = 101 \quad \psi_2(3214) = 001 \quad \psi_2(3241) = 001$$

$$\psi_2(3412) = 101 \quad \psi_2(3421) = 001$$

$$\psi_2(4123) = 110 \quad \psi_2(4132) = 100 \quad \psi_2(4213) = 010 \quad \psi_2(4231) = 010 \quad \psi_2(4312) = 100$$

$$\psi_2(4321) = 000$$

We have that

$$\rho_2(000) = \rho(111) = 1 \text{ (minimum parikh statistics)}$$

$$\rho_2(100) = \rho_2(011) = \rho_2(110) = \rho_2(001) = 3 \text{ (investigate most frequent parikh set statistic, in this case it's 3)}$$

$$\rho_2(010) = \rho_2(101) = 5 \text{ (maximum parikh statistics the alternating permutations contributes to this)}$$

parikh

000 – 4321

111 – 1234

001 – 3421, 3241, 3214

110 – 1243, 1423, 4123

100 – 4312, 4132, 1432

011 – 2134, 2314, 2341

010 – 2143, 2413, 2431, 4213, 4231

101 – 3412, 3142, 1342, 3124, 1324

010 and 101 can also be arranged as follows

010 – 2143, 2413, 2431, 4213, 4231

101 – 3412, 3123, 3124, 1342, 1324

descent

000 – 4321

111 – 1234
 001 – 4312, 3214, 4213
 011 - 2134, 4123, 3124
 100 – 3421, 1432, 2431
 110 – 1243, 2341, 1342
 010 – 4231, 3142, 3241, 4132, 2143
 101 – 1324, 2413, 2314, 1423, 3412

write something about this arrangement how both reversal and complement generate the same permutations and why (a proposition) and how the parikh set properties are affected by reversal and complement.

(symmetrin är intuition för att 010 och 101 innehåller fler antal permutationer).

Proposition1: There's a bijection $\bar{\phi}: P^{k-1} \rightarrow P^{k-1}$ defined by $\bar{\phi}(v) = \bar{v}$, where $\bar{v}$ is the complement vector of v), and where $v = \psi_2(\pi)$, $\pi \in S_k$

Proposition1 version2 Let $\pi = \pi_1 \cdots \pi_k \in S_k$. There's a bijection $\bar{\phi}: S_k \rightarrow S_k$ defined by $\bar{\phi}(\pi) = \bar{\pi}$ where $\bar{\pi} = \pi_k \cdots \pi_1$.

Proposition2: There's a bijection $\bar{\phi}: S_k \rightarrow S_k$ defined by $\bar{\phi}(\pi) = \bar{\pi}$ where $\bar{\pi}_i = k+1 - \pi_i$, $1 \leq i \leq k$

As a consequence of proposition1 and 2 we get that the composition of functions $\bar{\phi} \circ \bar{\phi}: S_k \rightarrow S_k$ is a bijection such that $(\bar{\phi} \circ \bar{\phi})(\pi) = \pi = (\phi \circ \bar{\phi})(\pi)$ and also, the parikh statistics for parikh vectors $\psi_2(\pi) = v$, $\psi_2(\bar{\pi}) = \bar{v}$, $\psi_2(\bar{\pi}) = \bar{v}$ and $\psi_2(\pi) = v = \psi_2(\pi)$.

Lemma $\rho(\psi_2(\pi)) = \rho(\psi_2(\bar{\pi})) = \rho(\psi_2(\bar{\pi})) = \rho(\psi_2(\pi)) = \rho(\psi_2(\pi))$.??

The same pattern is noted in the eulerian numbers concerning permutation statistics for parikh vectors of equal weight. We see that the bijective function $\bar{\phi}: S_k \rightarrow S_k$ leaves

$A(k, l+1) = A(k, k-l+1)$ invariant, where we have $\omega(\psi_2(\pi)) = l$ and $\omega(\psi_2(\bar{\pi})) = k-l$

We may ask ourselves is there a permutation statistic $\phi(\varphi(\pi))$ mapping that when considered by an analogical weight function will result in the eulerian number $A(k, l+1)$ but still will not have same distribution as $\rho(v)$?

Definition The *excedance set* is the set defined by $E(\pi) = \{i : \pi_i > i\}$, $\pi \in S_k, 1 \leq i \leq k-1$.

Definition The *excedance set statistic* is defined as $\varepsilon(T) = \#\{\pi \in S_k : E(\pi) = T\}$ where $T \subseteq [k-1]$.

Definition: The analog permutation statistic weight function of $\omega(\psi_2(\pi)) = |\psi_2(\pi)|$ and $d(\pi) = |D(\pi)|$ for excedance sets is defined by $e(\pi) = |E(\pi)|$

Proposition The following holds for the “weighted” permutation statistic for excedance sets $\#\{\pi \in S_k : e(\pi) = l\} = \#\{\pi \in S_k : \omega(\psi_2(\pi)) = l\} = \#\{\pi \in S_k : d(\pi) = l\} = A(k, l+1)$, where $A(k, l+1)$ is the eulerian number.

The excedance set statistic does not have the same distribution as the parikh or descent set statistic. Lets demonstrate this by a simple example. Lets take a look at $\bar{\phi}^{-1}(E(S_k))$.

Let $\pi \in S_4$. We have $\phi^{-1}(E(1234)) = \phi^{-1}(\{\emptyset\}) = 000$ (change to this notation for the rest),
 $E(1243) = \{3\} \rightarrow 001$, $E(1324) = \{2\} \rightarrow 010$,
 $E(1342) = \{2, 3\} \rightarrow 011$, $E(1423) = \{2\} \rightarrow 010$, $E(1432) = \{2\} \rightarrow 010$, $E(2134) = \{1\} \rightarrow 100$,
 $E(2143) = \{1, 3\} \rightarrow 101$, $E(2314) = \{1, 2\} \rightarrow 110$, $E(2341) = \{1, 2, 3\} \rightarrow 111$,
 $E(2413) = \{1, 2\} \rightarrow 110$, $E(2431) = \{1, 2\} \rightarrow 110$, $E(3124) = \{1\} \rightarrow 100$,
 $E(3142) = \{1, 3\} \rightarrow 101$, $E(3214) = \{1\} \rightarrow 100$, $E(3241) = \{1\} \rightarrow 101$, $E(3412) = \{1, 2\} \rightarrow 110$,
 $E(3421) = \{1, 2\} \rightarrow 110$, $E(4123) \rightarrow \{1\} \rightarrow 100$,
 $E(4132) \rightarrow \{1\} \rightarrow 100$, $E(4213) \rightarrow \{1\} \rightarrow 100$, $E(4231) \rightarrow \{1, 2, 3\} \rightarrow 101$,
 $E(4312) \rightarrow \{1, 2\} \rightarrow 110$, $E(4321) \rightarrow \{1, 2, 3\} \rightarrow 110$.

Lemma: $e(v) = e(\tilde{v}) = e(\check{v})$, where $\phi^{-1}(E(\pi)) = v$, and

Lemma2: If $\pi = \pi_1 \cdots \pi_k \in S_k$ and $\phi^{-1}(E(\pi_1 \pi_2 \cdots \pi_k)) = v \Leftrightarrow \phi^{-1}(E(\pi_k \pi_{k-1} \cdots \pi_1)) = \tilde{v}$ (define the bijection)

For Descent and parikh

If the vector is a palindrome there are only two permutations statistics of equal value for instance 101 and 010 while there are 4 for non-palindrome vectors for example 011 001

And for $e(P([k-1]))$ where $P([k-1])$ is the power set of $[k-1]$ for in this case $k = 4$. Let

$\phi: T \subseteq [k-1] \rightarrow \mathbb{Z}_2^{k-1}$ be defined as earlier. Then we have

$$e(\phi^{-1}(000)) = e(\{\emptyset\}) = 1 = e(\{1, 2, 3\}) = e(\phi^{-1}(111)),$$

$$e(\phi^{-1}(100)) = e(\{1\}) = 7 = e(\{1, 2\}) = e(\phi^{-1}(110)),$$

$$e(\phi^{-1}(010)) = e(\{2\}) = 3 = e(\{1, 3\}) = e(\phi^{-1}(101)),$$

$$e(\phi^{-1}(001)) = e(\{3\}) = 1 = e(\{2, 3\}) = e(\phi^{-1}(011)).$$

We see that (show eulerian number define excedance polynomial)

Notations, definitions, curiosities and results

By corollary we know that $P_{M_k}(M_{k+1}) = \{\pi \in S_k : \psi_{M_k}(\pi) = M_{k+1}\} =$

$P_2(v) = \{\pi \in S_k : \psi_2(\pi) = v \in P^{k-1} = \mathbb{Z}_2^{k-1}\}$. And we have

$\rho_{M_k}(M_{k+1}) = |P_{M_k}(M_{k+1})| = \rho_2(v) = |P_2(v)|$. Define a partial order on $P^{k-1} = \mathbb{Z}_2^{k-1}$ such that

$v_i \leq v_j$, if $\omega(v_1) \leq \omega(v_2)$. Define a partial order on P^{k-1} by $P_s^{k-1} \leq P_t^{k-1}$ if $|P_s^{k-1}| \leq |P_t^{k-1}|$.. This is a lattice...(fig. Hasse Diagram)

(proposition) Every mapping ϕ of a finite set A into the set of equivalence classes under ϕ is equivalent to a partition of A. (Proof in Abstract algebra)

Partition P^{k-1} into the sets P_l^{k-1} , where $P_l^{k-1} = \{v_i \in P^{k-1} : \omega(v_i) = l, 0 \leq l \leq k-1\}$. Lets denote the set of all sets P_l^{k-1} by $\forall P_l^{k-1}$. We have $P_s^{k-1} \cap P_t^{k-1} = \emptyset$ and $\forall P_l^{k-1} = \bigcup_{l=0}^{k-1} P_l^{k-1} = P^{k-1}$. $\forall P_l^{k-1}$ is a partitioning of P^{k-1} . Define a partial order on $\forall P_l^{k-1}$ by $P_s^{k-1} \leq P_t^{k-1}$ if $s \leq t$. Since $s = \omega(v_i) \leq \omega(v_j) = t, v_i \in P_s^{k-1}, v_j \in P_t^{k-1}$ we may say that $\forall P_l^{k-1} \leq_\omega$ is a partial order on $\forall P_l^{k-1}$ under ω . It's easy to see that this is a chain. It's easy to see that this is a chain.

(By proposition) Partitioning P^{k-1} into the sets P_l^{k-1} is the same as defining an equivalence relation on P^{k-1} such that $v_i \sim v_j \Leftrightarrow \omega(v_i) = \omega(v_j)$, resulting in the set of all equivalence classes of P^{k-1} under ω denoted by P^{k-1} / ω . Hence we have defined the set of equivalence classes P^{k-1} / ω of P^{k-1} under ω and the partially ordered set $P^{k-1}_{\leq_\omega}$ of P^{k-1} under ω .

We will later define a number $p(k, l)$, $1 \leq l \leq k$, where

$p(k, l) = \#\{\pi \in S_n : \psi_2(\pi) = v_i \in P_{l-1}^{k-1}\} = \#\{\pi \in S_n : \omega(\psi_2(\pi)) = \omega(v_i) = l-1\}$, Thus counting the number of permutations mapping to 2:nd order parikh vectors with weight $l-1$. By lemma this is the same as the number of permutations mapping to the parikh matrices of a certain weight (compute the weight). Later we will also see $p(k, l)$

is equal to the eulerian number $A(k, l+1) = \#\{\pi \in S_k : d(\pi) = k-1\}$ by first showing, among other things, that $\omega(v) = d(\pi)$. Define a vector $u_a = (p(k, 1), \dots, p(k, k))$. The weight of u_a is $\omega(u_a) = p(k, 1) + \dots + p(k, k) = k!$

Take a look at the chain $\forall P_l^{k-1} \leq_\omega = \{P_0^{k-1} \leq P_1^{k-1} \leq \dots \leq P_{k-1}^{k-1}\}$. We may define another curious vector $x = (x_1, \dots, x_k) \in P_0^{k-1} \times \dots \times P_{k-1}^{k-1}$ where the entries are vectors $v_i \in P_l^{k-1} \in P^{k-1} / \omega$, and

$$\omega(|x|) = \omega(|x_1|, \dots, |x_k|) = \omega(|P_0^{k-1}|, \dots, |P_{k-1}^{k-1}|) = |P_0^{k-1}| + \dots + |P_{k-1}^{k-1}| = \sum_{l=0}^{k-1} \binom{k-1}{l} = 2^{k-1} = |\forall P_l^{k-1}|$$

$$\begin{aligned} \text{While } |P_0^{k-1}| \dots |P_{k-1}^{k-1}| &= \prod_{l=0}^{k-1} \binom{k-1}{l} = \frac{1}{0!} \cdot \frac{(k-1)}{1!} \cdot \frac{(k-1)(k-2)}{2!} \cdot \frac{(k-1)(k-2)(k-3)}{3!} \dots \frac{(k-1)!}{(k-1)!} = \\ &= \frac{1 \cdot (k-1)^{k-1} (k-2)^{k-2} (k-3)^{k-3} \dots (2)^2 (1)^1}{(0!)^k (1!)^{k-1} (2!)^{k-2} (3!)^{k-3} \dots (k-2)!^2 (k-1)!^1} = |P_0^{k-1} \times \dots \times P_{k-1}^{k-1}|. \text{ Just a curiosity.} \end{aligned}$$

Define a polynomial with exponent $\omega(v)$ in analog of the eulerian polynomial such that

$$p_k(x) = \sum_{\pi \in S_k} x^{1+\omega(\pi)}. \text{ When we evaluate } p_k(x) \text{ at } x=1 \text{ we get}$$

$$p(1) = p(k, 1)1^1 + \dots + p(k, k)1^k = k! = \omega(u_a).$$

We have that $p(k, l) = \#\{\pi \in S_n : \psi_2(\pi) = v_i \in P_{l-1}^{k-1}\} = \#\{\pi \in S_n : \omega(\psi_2(\pi)) = \omega(v_i) = l-1\}$. Lets denote the set $\{\pi \in S_n : \psi_2(\pi) = v_i \in P_{l-1}^{k-1}\}$ by $\wp_{l-1}$. Since we easily were able to define a chain $\forall P_l^{k-1} \leq_\omega = \{P_0^{k-1} \leq P_1^{k-1} \leq \dots \leq P_{k-1}^{k-1}\}$ we are able to construct a ordered polynomial

$p_k(x) = \sum_{\pi \in S_k} x^{1+\omega(\pi)} = \sum_{l=1}^k p(k,l)x^l$. Let the vector $v_i \in P^{k-1}$ be denoted by v_i^l is $v_i \in P_l^{k-1}$, that

is if $\omega(v_i) = l$. We want to construct a polynomial $\rho_k(x) = \sum_{\pi \in S_k} x^{1+\phi(\pi)} = \sum_{i=1}^l \rho(k,i)x^i$ where the

exponent is $\phi(\pi) = i$ (rank function) for some function ϕ such that $P_{l \leq \phi}^{k-1}$ is a chain

$\{v_1^l < v_2^l < \dots < v_{\binom{k-1}{l}}^l\}$ on P_l^{k-1} under ϕ . We have that the partial order $v_i^l \leq v_j^l$ if $\phi(v_i^l) \leq \phi(v_j^l)$

defined by **(find the function)**. We may extend this by defining a partial order $P_{l \leq \phi \circ \omega}^{k-1}$ on P^{k-1} under $\phi \circ \omega$. This poset will be a chain.

$$P_{l \leq \phi \circ \omega}^{k-1} = \{(v_1^0) < (v_1^1 < \dots < v_{\binom{k-1}{k-1}}^1) < (v_1^2 < \dots < v_{\binom{k-1}{k-1}}^2) < \dots < (v_1^{k-2} < \dots < v_{\binom{k-1}{k-2}}^{k-2}) < (v_{\binom{k-1}{k-1}}^{k-1})\}.$$

$$p(k,l) = \#\{\pi \in S_k : \psi_2(\pi) = v_i \in P_{l-1}^{k-1}\} = \#\{\pi \in S_k : \psi_2(\pi) = v_i^{l-1}, 1 \leq i \leq \binom{k-1}{l-1}\} = \sum_{i=1}^{\binom{k-1}{l-1}} \rho(v_i^l)$$

Now we may order the terms in $\rho_k(x)$. We have

$$\begin{aligned} \rho_k(x) &= \sum_{m=0, l=0}^{m=2^{k-1}, l=k-1} \left(\sum_{i=1}^{\binom{k-1}{l}} \rho(v_i^l) \right) x^{1+m} = \rho(v_1^0)x + \sum_{i=1}^{k-1} \rho(v_i^1)x^{\binom{k-1}{0}+i} + \sum_{i=1}^{\binom{k-1}{2}} \rho(v_i^2)x^{\binom{k-1}{0}+\binom{k-1}{1}+i} + \\ &+ \sum_{i=1}^{\binom{k-1}{3}} \rho(v_i^3)x^{\sum_{j=0}^2 \binom{k-1}{j}+i} + \dots + \sum_{i=1}^{\binom{k-1}{k-1}} \rho(v_i^{k-1})x^{\sum_{j=0}^{k-2} \binom{k-1}{j}+i}. \end{aligned}$$

When $\rho_k(x)$ is evaluated at $x=1$, we get

$$\begin{aligned} \rho_k(1) &= \sum_{m=0, l=0}^{m=2^{k-1}, l=k-1} \left(\sum_{i=1}^{\binom{k-1}{l}} \rho(v_i^l) \right) 1^{1+m} = \rho(v_1^0) + \sum_{i=1}^{k-1} \rho(v_i^1) + \sum_{i=1}^{\binom{k-1}{2}} \rho(v_i^2) + \sum_{i=1}^{\binom{k-1}{3}} \rho(v_i^3) + \dots + \sum_{i=1}^{\binom{k-1}{k-1}} \rho(v_i^{k-1}) = \\ &= p(k,1)1^1 + \dots + p(k,k)1^k = p_k(1) = \sum_{l=1}^k p(k,l)1^l = \sum_{\pi \in S_k} 1^{1+\omega(\pi)} = k! \end{aligned}$$

We have that $|P_l^{k-1}| = \binom{k-1}{l}$ and $|P^{k-1}| = |\vee P_l^{k-1}| = \sum_{l=0}^{k-1} \binom{k-1}{l} = 2^{k-1}$.

Define a partial order $P_{l \leq}^{k-1}$ every set P_l^{k-1} such that it is a chain (totally ordered poset).

That is we need to find a function ϕ such that. Then combine this with the prior partial order to get a chain of the whole of P^{k-1} .

Now if we take a look at $\psi_2(\pi)$ we may observe that ... (take the rules of complement and retrograd equivalence relations) Take into account that $\psi_2(\pi)$ is a surjective function mapping to $k-1$ binary strings. Define parikh statistic and compute eulerian polynomial and statistics. Conjecture that there is a bijection. Proove it. Show consequences.

Parikh

1111 – 12345
 0000 - 54321
 0001 – 45321,43521,43251,43215
 1110 – 12354,12534,15234,51234
 1000 – 54312,54132,51432,15432
 0111 – 21345,23145,23415,23451
 1100 - 12543,51243,51423,54123,15243,15423
 0011 – 34521,43215,32415,32145,34251,32451
 1101 – 12435,12453,14235,14253,14523,41235,41253,41523,45123
 0010 – 53421,35421,53241,35241,32541,53214,35214,32514,32154
 1011 – 13245,31245,13425,31425,34125,13452,31452,34152,34512
 0100 – 54231,54213,52431,52413,52143,25431,25413,25143,21543
 1001 – 41325,41325,41532,43125,43152,43512,45132,45312,14325,14352,14532
 0110 – 52314,52314,23514,52134,25134,21534,23154,21354,52341,25341,23541

 1010 - 13254,13524,13542,31254,31524,31542,35124,35142,35412,51324,51342,
 53124,53142,53412,15324,15342

 0101- 45231,42531,24531,45213,42513,24513,42153,24153,21453,42315,24315
 42135,24135,21435,42351,24351

den tredje diagonalen är parikh vektorn av ordning 2. Låt oss notera detta. Parikh statistiken för parikh vektorer av 2:a ordningen defineras som antalet ord som går på en given parkh vektor av ordning 2.

$\Psi_2(w) = (|w|_{a_{1,2}}, |w|_{a_{2,3}}, \dots, |w|_{a_{(n-1),n}})$ the length of $\Psi_2(w)$ is $\ell(\Psi_2(w)) = n - 2 + 1 = n - 1$

En parikh matris av ordning k $\Psi_k(w) = (|w|_{1,k}, |w|_{2,k+1}, \dots, |w|_{(n-1),n})$ we have

$\ell(\Psi_k(w)) = n - k + 1$. Låt $\Sigma = \{1 < 2 < \dots < n\}$ vara det ordnade alfabetet av $[n]$

Om $w \in S_n(\Sigma)$ så är $S_n(\Sigma) = S_n$. Vi betraktar alltså inte alla ord $\Sigma^* = S_n(M)$

(permutationer av multimängder) utan vanliga permutationer av mängder.

Då är parikh matriserna binära och parikh vektorn

$\Psi_k(w) = (|w|_{1,k}, |w|_{2,k+1}, \dots, |w|_{(n-1),n}) \in P_2^{n-1} \subseteq \mathbb{N}_2^{n-1}$.

Definiera vikten av parikh vektorerna som $\omega(\Psi_k(w)) = |w|_{1,k} + |w|_{2,k+1} + \dots + |w|_{(n-1),n}$

Vikten av parikh matrisen $\psi_{M_k}(w)$ definieras som $\omega(\psi_{M_k}(w)) = \sum_{k=1}^n \omega(\Psi_k(w))$

Definition: The let $w \in \Sigma^* = S_n(M)$ parikh vector set is defined as

$P_k(w) = \{w \in \Sigma^* : \psi_k(w) = v \in P_k^{n-k+1}\}$. The parikh matrix set is defined as

$P_{M_k}(w) = \{w \in \Sigma^* : \psi_{M_k}(w) = M_k\}$

Definition: The parikh vector statistic is defined as the number of words mapping to a given parikh vector, the parikh matrix statistic is defined as the number of parikh matrices mapping to a given parikh vector. Let $v \in P_k^{n-k+1} \subseteq \mathbb{N}_k^{n-k+1}$ then this is denoted

$\rho_k(v) = \#\{w \in S_n(M) : \psi_k(w) = v\}$ and $\rho_M(v) = \#\{w \in S_n(M) : \psi_{M_k}(w) = M_k\}$

Is it possible to define a statistic such that $\mathcal{G}_k(v) = \#\{w \in S_n(M) : \psi_k(w) \subseteq v \in P_k^{n-k+1}\}$ and

$\mathcal{G}_{M_k}(v) = \#\{w \in S_n(M) : \psi_{M_k}(w) \subseteq M_k\}$.

Note that if we have a multiset $M = \{1^{a_1}, 2^{a_2}, \dots, m^{a_m}\}$, That is $N = \{1, 2, \dots, m\}$ and a function v

such that $v(1) = a_1, v(2) = a_2, \dots, v(m) = a_m$, and $\#M = \sum_{x \in N} v(x) = \sum_{i=1}^m a_i = n$.

Definition define a subvector u of v in P_k^{n-k+1} by $u \leq v$ if $u = (|w|_{b_{1,k}}, |w|_{b_{2,k+1}}, \dots, |w|_{b_{n-k+1,n}})$ and

$v = (|w|_{a_{1,k}}, |w|_{a_{2,k+1}}, \dots, |w|_{a_{n-1,n}})$ with $|w|_{b_{1,k}} \leq |w|_{a_{1,k}}, \dots, |w|_{b_{n-k+1,n}} \leq |w|_{a_{n-k+1,n}}$ (visa partiell ordning dvs reflexiv, transitiv och antisymmetrisk).

Kanske vi kommer ha $\rho_k(v) = \sum_{u \leq v} \mathcal{G}_k(u)$ och $\rho_k(M_k) = \sum_{N_k \leq M_k} \mathcal{G}_k(N_k)$ om vi kan visa

hypotesen. Ett steg att visa den är att visa att $\rho_k(v) = \sum_{u \leq v} \mathcal{G}_k(u)$ dessutom.

Om vi betraktar specialfallet ovan då w är ord av permutationer $w = \pi \in S_5$ har vi

Att betrakta $\rho_2(v) = \#\{\pi \in S_5 : \psi_2(\pi) = v \in P_2^{n-1}\}$. Dvs

$$\rho(1111) = 1$$

$$\rho(1110) = 4$$

$$\rho(1101) = 9$$

$$\rho(1011) = 9$$

$$\rho(0111) = 4$$

$$\rho(1100) = 6$$

$$\rho(1001) = 11$$

$$\rho(0110) = 11$$

$$\rho(0011) = 6$$

$$\rho(1010) = 16$$

$$\rho(0101) = 16$$

$$\rho(1000) = 4$$

$$\rho(0100) = 11$$

$$\rho(0010) = 11$$

$$\rho(0001) = 4$$

$$\rho(0000) = 1$$

We have $D(\pi) = \{i : a_i > a_{i+1}\}$ and $d(\pi) = \# D(\pi)$. $A_n(x) = \sum_{\pi \in S_n} x^{1+d(\pi)}$ is called the *Eulerian polynomial*. The coefficients of $A_n(x)$, denoted $A(n, k)$ are known as the *Eulerian numbers*. We have $A(n, k) = \# \{\pi \in S_n : d(\pi) = k-1\}$.

$$A_4(x) = x + 11x^2 + 11x^3 + x^4 \text{ and } x + 26x^2 + 66x^3 + 26x^4 + x^5.$$

$$\text{We have } \sum_{T \subseteq S} \beta(T) = \sum_{i=1}^k A(n, i) = n!,$$

$$\text{and we would expect, by the hypothesis that } \sum_{\omega(v)=k} \rho(v) = A(n, k) \Leftrightarrow \sum_{|S|=k} \beta(S) = A(n, k).$$

$$\sum_{\omega(v)=0} \rho(v) = 1, \quad \sum_{\omega(v)=1} \rho(v) = \rho(1000) + \rho(0100) + \rho(0010) + \rho(0001) = 4 + 9 + 9 + 4 = 26.$$

$$\sum_{\omega(v)=2} \rho(v) = \rho(0101) + \rho(1010) + \rho(0011) + \rho(0110) + \rho(1100) + \rho(1001) = 16 + 16 + 6 + 11 + 11 + 6 = 66$$

$$\sum_{\omega(v)=3} \rho(v) = \rho(0111) + \rho(1011) + \rho(1101) + \rho(1110) = 4 + 9 + 9 + 4 = 26.$$

$$\sum_{\omega(v)=4} \rho(v) = \rho(1111) = 1$$

Lets check the **descent** set mapping in $\phi^{-1}(D(S_k)), k=5$.

1111 – 12345

0000 – 54321

1110 – 12354, 12453, 13452, 23451

0001 – 54312, 54213, 53214, 43215

0111 – 21345, 31245, 41235, 51234

1000 – 45321, 35421, 25431, 15432

1100 – 34521, 14532, 23541, 24531, 12543, 13542

0011 – 32145, 52134, 43125, 42135, 54123, 53124 (complement vector gives complement perm.)

0010 – 32154, 42153, 43152, 43251, 52143, 53142, 53241, 54132, 54231

1101 – 34512, 24513, 23514, 23415, 14523, 13524, 13425, 12534, 12435

0100 – 21543, 31542, 41532, 51432, 32541, 42531, 52431, 43521, 53421

1011 – 45123, 35124, 25134, 15234, 34125, 24135, 14235, 23145, 13245

1001 – 34215, 35214, 35412, 45213, 45312, 14325, 15324, 15423, 25314, **25413**, 24315

0110 – 51243, 41253, 21453, 31254, 21354, 52341, 42351, 32451, 41352, 31452, 51342

0101 – 53412, 41325, 41523, 42315, 42513, 43512, 51324, 51423, 52314, 52413, 21435, 21534, 31425, 31524, 32415, 32514

1010 – 13254, 25341, 25143, 24351, 24153, 23154, 15342, 15243, 14352, 14253, 45231, 45132, 35241, 35142, 34251, 34152

Parikh

1111 – 12345

0000 – 54321

0001 – 45321,43521,43251,43215
 1110 – 12354,12534,15234,51234
 1000 – 54312,54132,51432,15432
 0111 – 21345,23145,23415,23451
 1100 – 12543,51243,51423,54123,15243,15423
 0011 – 34521,43215,32415,32145,34251,32451
 1101 – 12435,12453,14235,14253,14523,41235,41253,41523,45123
 0010 – 53421,35421,53241,35241,32541,53214,35214,32514,32154
 1011 – 13245,31245,13425,31425,34125,13452,31452,34152,34512
 0100 – 54231,54213,52431,52413,52143,25431,25413,25143,21543
 1001 – 41325,41325,41532,43125,43152,43512,45132,45312,14325,14352,14532
 0110 – 52314,52314,23514,52134,25134,21534,23154,21354,52341,25341,23541
 1010 – 13254,13524,13542,31254,31524,31542,35124,35142,35412,51324,51342,
 53124,53142,53412,15324,15342
 0101- 45231,42531,24531,45213,42513,24513,42153,24153,21453,42315,24315
 42135,24135,21435,42351,24351

Yamba's hypothesis second (correct) version

Let $\phi: P(S) \rightarrow \mathbb{Z}_2^{k-1}$, $S \subseteq [k-1] \mapsto v \in \mathbb{Z}_2^{k-1}$ be the function defined as above. where
 $\phi^{-1}(v) = S$ (show prior to this that ϕ is a bijection), $S = D(\pi)$, $\pi \in S_k$ and $v = \psi_2(y)$, $y \in S_k$.
 Then $\forall v \in \mathbb{Z}_2^{k-1}$, $\forall S \in P(S)$ the following holds $\rho(v) = \beta(S)$.

Yamba's hypothesis second symbolic (correct) version

Let $S \subseteq [k-1]$ and $v = \phi^{-1}(S)$. Then

$$\rho(v) = \#\{y \in S_k : \psi_2(y) = v\} = \#\{\pi \in S_k : D(\pi) = S\} = \beta(S)$$

(The parikh statistic distribution is equal to the descent statistic distribution when we work with words in S_k).

Proof: Define a partial order on $\mathbb{Z}_2^{k-1}$ such that $u \leq v$ if $\phi^{-1}(u) \subseteq \phi^{-1}(v)$. It's easy to see that this is a reflexive, anti-symmetric and transitive relation. Define the inclusive permutation statistic by $\mathcal{G}(v) = \#\{y \in S_k : \psi_2(y) \leq v \leq \sum_{k-1} 1 \in \mathbb{Z}_2^{k-1}\}$ and $\rho(v) = \#\{y \in S_k : \psi_2(y) = v\}$.

Clearly $\mathcal{G}(v) = \sum_{u \leq v} \rho(u)$ so if we can find a formula for this and invert it we'll get the desired

result. Let $v = v_1 \cdots v_{k-1} \leq \sum_{k-1} 1 \in \mathbb{Z}_2^{k-1}$, such that $\phi^{-1}(v_1 \cdots v_{k-1}) = S = \{s_1, \dots, s_j\}$ where

$v_i = 1 \Leftrightarrow s_i \in S$. Note that s_i denotes an index for which v has a 1, all the others 0. Since

$\psi_2(y) = v$ is a parikh vector of second order this means that v at index s_i has the entry

$v_{s_i} = |\lambda|_{s_i, s_i+1} = 1$ and all others before s_i is 0 which means for all $l < i$ we have

$s_{l-1, l} = 0 \Leftrightarrow s_{l, l-1} = 1$. Therefore $s_{l, l}$ must exist so to obtain the permutation $y = y_1 y_2 \cdots y_k \in S_k$

satisfying $\psi_2(y) \leq v$ we first placing the following into left to right order as a scattered

subword $s_1(s_1 - 1) \cdots 1$ in $\mathcal{Y}$ with $s_1 > s_1 - 1 > \cdots > 1$ (since the descent statistic is inclusive $s_1 + 1$ is not taken into account. $|\mathcal{Y}|_{s_1, s_1+1} = 1$ or 0) among k available positions.

That is assign $\lambda_{s_1} = s_1, \lambda_{s_1-1} = s_1 - 1, \dots, \lambda_1$ such that $\lambda_{s_1} \lambda_{s_1-1} \cdots \lambda_1$ is a scattered subword of

$\lambda_1 \cdots \lambda_k$ there are ofcourse $\binom{k}{s_1}$ such subwords. Then place $s_2 > s_2 - 1 > \cdots > s_1 + 1$ as a

scattered subword in $\mathcal{Y}$ among the $k - s_1$ available positions. Having already placed

$s_1 > s_1 - 1 > \cdots > 1$ as a scattered subword in $\mathcal{Y}$ assigning $\lambda_{s_2} = s_2, \lambda_{s_2-1} = s_2 - 1, \dots, \lambda_{s_1+1} = s_1 + 1$

can be done in $\binom{k - s_1}{s_2 - s_1}$ ways and so on. Therefore we have that

$$\mathcal{G}(v) = \binom{k}{s_1} \binom{k - s_1}{s_2 - s_1} \binom{k - s_2}{s_3 - s_2} \cdots \binom{k - s_j}{k - s_j} = \binom{k}{s_1, s_2 - s_1, s_3 - s_2, \dots, k - s_j}.$$

Now recall that we have $\mathcal{G}(v) = \sum_{u \leq v} \rho(u)$. Hence by the principle of inclusion exclusion

$$\text{principle (the formula) we get } \rho(v) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_t \leq j} (-1)^{j-t} \binom{k}{s_{i_1}, s_{i_2} - s_{i_1}, \dots, k - s_{i_t}} = \beta(S). \quad \square$$

In other words this means that parikh statistics has the same distribution as the Descent statistics on permutations of sets. Parikh vectors are descent sets in some other form. Exactly how the two interrelate is open to further exploration but first a proof of the hypothesis is vital.

A few more observations:

Parikh mapping structure

Lemma: Let $\bar{\gamma}, \bar{\gamma}, \bar{\gamma}, \bar{\gamma} \in S_k$ and $\bar{v}, \bar{v}, \bar{v}, \bar{v} \in \mathbb{Z}_2^{k-1}$ are parikh vectors of second order.

If $\psi_2(\gamma) = v$, then $\psi_2(\bar{\gamma}) = \bar{v}, \psi_2(\bar{\gamma}) = \bar{v}$ and $\psi_2(\bar{\gamma}) = \bar{v}$.

Proof

Descent mapping structure

Lemma: Let $\bar{\pi}, \bar{\pi}, \bar{\pi}, \bar{\pi} \in S_k$ and $\bar{v}, \bar{v}, \bar{v}, \bar{v} \in \mathbb{Z}_2^{k-1}$ where $v = \phi^{-1}(S), \bar{v} = \phi^{-1}(\bar{S}), \bar{v} = \phi^{-1}(\bar{S})$ and $\bar{v} = \phi^{-1}(\bar{S})$. If $\phi^{-1}(D(\pi)) = v$, then $\phi^{-1}(D(\bar{\pi})) = \bar{v}, \phi^{-1}(D(\bar{\pi})) = \bar{v}$ and $\phi^{-1}(D(\bar{\pi})) = \bar{v}$.

Proof

We see that the mapping structure of descent mapping differs from the structure of the parikh mapping. Even though I conjectured the parikh statistic and the descent statistic to be equivalent we should not expect one permutation having a certain descent set to be in the corresponding parikh vector of second order. It can be showed that we can find a permutation with a descent set who's descent statistic is k but the same permutation may correspond to a

parikh vector differing from the descent set and who's parikh statistic differs from k . Still the distribution of the parikh and descent statistic is equal even though the permutations contributing to the statistic differs between the two.

Parikh structure equivalence

Lets define an equivalence relation on $\mathbb{Z}_2^{k-1}$ by $u \sim v$ if $\bar{\phi}(u) = \bar{u} = v$. Where u and v are parikh vectors of second order. That is $\psi_2(\lambda) = v$ and $\psi_2(\bar{\lambda}) = u$. So in effect $u \sim v$ if $\bar{\phi}(u) = \bar{u} = v$ implies $\lambda \sim \gamma$ if $\bar{\lambda} = \gamma$. But for enumerative convinience we will foremost consider the equivalence on parikh vectors and let permutations be distinct. Lets denote the set of equivalence classes as defined above on $\mathbb{Z}_2^{k-1}$ under $\bar{\phi}$ by $\mathbb{Z}_2^{k-1} / \bar{\phi}$. Equivalence classes may be denoted by their a representative vector $[v]_{\bar{\phi}}$ such that $\omega(v) \leq \omega(\bar{v})$ and if $\omega(v) = \omega(\bar{v})$ choose the vector whos first entry is 0.

Example: Take the parikh vector mapping $\psi_2(S_4)$:

000 – 4321
 111 – 1234
 001 – 3421, 3241, 3214
 110 – 1243, 1423, 4123
 100 – 4312, 4132, 1432
 011 – 2134, 2314, 2341
 010 – 2143, 2413, 2431, 4213, 4231
 101 – 3412, 3142, 1342, 3124, 1324

This will collapse into the following mapping

$$\psi_2(S_4) / \bar{\phi}$$

000 – 4321, 1234
 001 – 3421, 3241, 3214, 1243, 1423, 4123
 100 – 4312, 4132, 1432, 2134, 2314, 2341
 010 – 2143, 2413, 2431, 4213, 4231, 3412, 3142, 1342, 3124, 1324

Here each parikh set contain permutations togheter with their reverseals. Thus if we define a parikh statistic on these equivalence vectors we see that $\rho([v]) = 2\rho(v)$. Now lets define a second equivalence relation where $u \sim v$ if $\bar{\phi}(u) = \bar{u} = \bar{v}$. $u \sim v$ if $\bar{\phi}(u) = \bar{u} = \bar{v}$ implies that $\lambda \sim \gamma$ if $\bar{\lambda} = \gamma$ since $\psi_2(\lambda) = u$ and $\psi_2(\bar{\lambda}) = \psi_2(\gamma) = v$. If we consider the set of equivalence classes on $\mathbb{Z}_2^{k-1} / \bar{\phi}$ under $\bar{\phi}$ denoted by $\mathbb{Z}_2^{k-1} / \bar{\phi} \circ \bar{\phi}$ the $\psi_2(S_4) / \bar{\phi}$ mapping would collapse into the following. If we let the vector containing the first leftmost 0 be the representative equivalence class element $[v]_{\bar{\phi} \circ \bar{\phi}}$:

$$\psi_2(S_4) / \bar{\phi} \circ \bar{\phi}$$

000 – 4321, 1234
 001 – 3421, 3241, 3214, 1243, 1423, 4123, 4312, 4132, 1432, 2134, 2314, 2341
 010 – 2143, 2413, 2431, 4213, 4231, 3412, 3142, 1342, 3124, 1324

This leads us to the following lemmas

Lemma1: $\rho([v]_{\bar{\phi} \circ \phi}) = 2\rho([v]_{\bar{\phi}}) = 4\rho(v)$ if v is a non-palindrome such that $\bar{\phi}(v) = \bar{v} \neq v = \phi(v)$.

Lemma2: If v is a palindrome or if v satisfies $\bar{\phi}(v) = \bar{v} = v = \phi(v)$ there are exactly 2 vectors v and u satisfying $\rho(v) = \rho(u)$ and these are equivalent under $\bar{\phi}$.

By the Yamba hypothesis these results can be translated for descent statistics and we

$\rho([v]_{\bar{\phi} \circ \phi}) = \beta([S]_{\bar{\phi} \circ \phi})$ where $\phi(S) = v$. Now may formulate a polynomial whose coefficients are

$\rho([v]_{\bar{\phi} \circ \phi}) = \beta([S]_{\bar{\phi} \circ \phi})$, but first we need to define a partial order on the vectors that is a chain.

This is not difficult because it can be shown that the parikh statistics on the equivalence set classes consists of distinct values. In other words the corresponding parikh polynomial would have unique coefficients so they can be totally ordered.

Lemma Consider $\psi_2(S_k)$ The parikh polynomial under the equivalence relation $\bar{\phi} \circ \phi$ denoted

$\rho_{k, \bar{\phi} \circ \phi}(x) = \sum_{i=1}^{2^{k-2} - \zeta(\mathbb{Z}_2^{k-1})} \rho([v_i])x^i$, where $\zeta(\mathbb{Z}_2^{k-1})$ is proportional to the number of non-palindrome vectors.

Theorem(Yamba hypothesis 1)(false version)

Let D denote the set of all descent sets of permutations $\pi \in S_n$ and let $P(\psi_2(\pi))$ denote the set of all parikh vectors of order 2 of permutations $\pi \in S_n$. There is a bijection

$\phi: D \rightarrow P(\psi_2(\pi)), D(\pi) \mapsto \psi_2(\pi)$.

Proof (proof of bijection between powerset of $[k-1]$ and binary vectors of length $k-1$, not the correct proof of the correct yamba hypothesis) : Let $D(\pi) = \{s_1, s_2, \dots, s_j\}$ such that

$s_1 < s_2 < \dots < s_j, 1 \leq j \leq k$. Define a mapping $\phi: D \rightarrow P(\psi_2(\pi))$ by $\phi(S) = (e_1, \dots, e_{n-1})$, where

$$e_i = \begin{cases} 1 & \text{if } s_i \in S \\ 0 & \text{if } s_i \notin S \end{cases}, 1 \leq i \leq n-1.$$

$|D| = \sum_{S \subseteq [n-1]} |S| = 2^{n-1}$ and parikh vectors of second order are the 2^{n-1} vectors of $\mathbb{Z}_2^{n-1}$.

Since D and $P(\psi_2(\pi))$ have same cardinality we need only to show injectivity or surjectivity.

Lets assume that $\phi(S) = (e_1, \dots, e_{n-1})$, $\phi(T) = (d_1, \dots, d_{n-1})$ and $\phi(S) = \phi(T)$ then we have

$$e_1 = d_1, \dots, e_{n-1} = d_{n-1} \text{ and so by the definition of } \phi \text{ with } e_i = \begin{cases} 1 & \text{if } s_i \in S \\ 0 & \text{if } s_i \notin S \end{cases}, S = T, \text{ so } \phi \text{ is}$$

injective. Thus $\phi: D \rightarrow P(\psi_2(\pi))$ is a bijection. $\square$

(Alternative Proof) Show that two functions between the same sets with the same distribution are in bijection. Same distribution meaning the same number of elements in the domain mapping to a given element in the codomain for all elements involved.

Show that the descent statistic has the same distribution (cardinalities) as the parikh statistic.

If we take the sum of vectors with equal weight we get $\sum_{v_l \in P_l^{k-1}} \rho(v_l) = A(k, l+1) = \sum_{S: |S|=l, S \subseteq [k-1]}$

Corollary: $\phi: D \rightarrow P(\psi_2(\pi))$ be as defined above. The parikh vector statistic of second order $\rho_k(v) = \#\{w \in S_n(M) : \psi_k(w) = v\}$ is given by the formula $\rho_k(v) = \beta(\phi^{-1}(v)) = \beta_n(S)$.

Proof: From the bijection $\phi: D \rightarrow P(\psi_2(\pi))$, we have that $\rho_k(v) = \#\{w \in S_n(M) : \psi_k(w) = v\} = \#\{\pi \in S_n : \phi^{-1}(\psi_k(\pi)) = \phi^{-1}(v) = S\} = \#\{\pi \in S_n : D(\pi) = S\} \square$

The next big step would be to generalize the Yamba hypothesis to all words, that is for all permutations of multisets.

Notations. Let N be a set $N = \{1, 2, \dots, k\}$. The natural alphabet of N is $\Sigma_k = \{1 \leq 2 \leq \dots \leq k\}$.

Note that there are $k!$ alphabets Σ_k of N . $k!-1$ are non natural and the naturals are considered as letters with a predefined order.

The words with letters from the alphabet Σ_k is denoted Σ_k^* . The words of length n from the alphabet Σ_k is denoted $\Sigma_{n,k}^*$. If Σ_k is an alphabet of the set $N = \{1, 2, \dots, k\}$ and

$M = \{1^{a_1}, 2^{a_2}, \dots, k^{a_k}\}$ is a multiset of n . then the words of length n with letters from the alphabet Σ_k of the set N is denoted $S_n(M)$, that is $\Sigma_{n,k}^* = S_n(M)$.

Definition: w is a palindrome iff $w = \overleftarrow{w}$

(Proposition) If $w = w_1 \dots w_n$ is a palindrome word, $w_i = w_{n-i+1}, 1 \leq i \leq n$

Proposition: $S_n(M)$ contains words that are palindromes $\Leftrightarrow M = \{1^{a_1}, 2^{a_2}, \dots, k^{a_k}\}$ is a multiset such that $a_i = 2b_i$, for all i except at most one where $a_j = 1, j \neq i$.

Proposition: Let $M = \{1^{a_1}, 2^{a_2}, \dots, k^{a_k}\}$ be a multiset such that $a_i = 2b_i$, for all i except at most one where $a_j = 1, j \neq i$, then

The number of palindrome words in $S_n(M)$ is $\binom{\lfloor n/2 \rfloor}{b_1, b_2, \dots, b_k}$

Lemma1: The number of palindrome-free words in $S_n(M)$ is even

Proof

Lemma2 The number of palindrome words in $S_n(M)$, $n > 3$ is even??

Proof 1

$$\binom{\lfloor n/2 \rfloor}{b_1, b_2, \dots, b_k} = \binom{\left\lfloor \frac{a_1 + \dots + a_k}{2} \right\rfloor}{b_1, b_2, \dots, b_k} = \binom{\left\lfloor \frac{2b_1 + \dots + 2b_k}{2} \right\rfloor}{b_1, b_2, \dots, b_k} = \binom{\lfloor b_1 + \dots + b_k \rfloor}{b_1, b_2, \dots, b_k} = \frac{(b_1 + \dots + b_k)!}{(b_1)! \dots (b_k)! \dots}$$

Proof 2 combinatorial. If n is even construct a palindrome by $w = w_1 \dots w_n$ such that

w can be factored into $w = xy$ such that $x = w_1 \cdots w_{n/2}$ and $y = w_{(n/2)+1} \cdots w_n$ with $x, y \in S_{n/2}(M_b)$ where $M_b = \{1^{b_1}, \dots, k^{b_k}\}$, $b_1 + \cdots + b_k = n/2$ by choosing a word x and y such that $y = \tilde{x}$. Hence $w = xy = x\tilde{x}$ is a palindrome. Given any palindrome word $x\tilde{x}$ there is another word palindrome word $\tilde{x}\tilde{\tilde{x}}$ (note that $\tilde{x\tilde{x}} \neq x\tilde{x}$). Therefore there are

$$|S_{n/2}(M_b)| = \left| \frac{a_1 + \cdots + a_k}{2} \right|_{b_1, b_2, \dots, b_k} = \left| \frac{2b_1 + \cdots + 2b_k}{2} \right|_{b_1, b_2, \dots, b_k} = \left| \frac{b_1 + \cdots + b_k}{b_1, b_2, \dots, b_k} \right| = \frac{(b_1 + \cdots + b_k)!}{(b_1)! \cdots (b_k)!}. \text{ Now proof that } \frac{(b_1 + \cdots + b_k)!}{(b_1)! \cdots (b_k)!} \text{ is even.}$$

(proposition) The set of parikh vectors is a poset (lattice)

Proof: (define a partial order by $<$)

(reflexivity, antisymmetry, transitivity)

Investigate what type of lattice. Distributive?

Draw examples with Hasse diagrams. Introduce other graphs and diagrams to show relations, with other relations than comparable $\rightarrow$ edge. Edge $\rightarrow$ related elements, nodes elements.

Maybe we need to generalize the principle of inclusion exclusion so that it incorporates multisets before we can solve the task of generalizing the Yamba hypothesis to $S_n(M)$.

Proposition: Let P^j be the set of parikh vectors of order j . Define the maximal parikh vector $\psi_j(w) = v = (v_1, \dots, v_{k-j+1})$ of P^j to be a vector such that it satisfies $u, v \in P^j$, $\psi_j(w)$, $u_i \leq v_i$, $1 \leq i \leq k-j+1$. Let $\sum_{n,k}^* = S_n(M)$. Then the maximal parikh vector of P^j is $\psi_j(w) = v$, where $w = w_1 \cdots w_n$ is the unique word where $w_i \leq w_{i+1}$, that is the ascending word in $S_n(M)$.

The maximal vector is $v = (|w|_1 \cdot |w|_2 \cdots |w|_j, |w|_2 \cdot |w|_3 \cdots |w|_{j+1}, \dots, |w|_{k-j+1} \cdot |w|_{k-j+1} \cdots |w|_k)$.

Corollary1: The maximal weight of the parikh vector of vectors in P^j as above is

$$\omega(v) = |w|_1 \cdot |w|_2 \cdots |w|_j + |w|_2 \cdot |w|_3 \cdots |w|_{j+1} + \cdots + |w|_{k-j+1} \cdot |w|_{k-j+1} \cdots |w|_k$$

Corollary2: The maximal parikh v vector in P^2 of second order is

$$v = (|w|_1 \cdot |w|_2, |w|_2 \cdot |w|_3, \dots, |w|_{k-1} \cdot |w|_k)$$

Proposition: Let $w \in \sum_{n,k}^* = S_n(M)$ then $\psi_2(w) + \psi_2(\tilde{w}) = (|w|_1 \cdot |w|_2, |w|_2 \cdot |w|_3, \dots, |w|_{k-1} \cdot |w|_k)$.

That is the vector sum of the parikh vector of second order of w and the same of its retrograde is equal to the maximal parikh vector of second order.

Proof

Lemma There is a bijection between sub multisets and parikh vectors of different orders.

Question Is $\binom{n}{a_1, \dots, a_k}$ even. If not for what $a_1, \dots, a_k$ is $\binom{n}{a_1, \dots, a_k}$ even?

Proposition Let $w \in S_n(M)$ be a palindrome. Then $\psi_2(w) = \frac{1}{2}(|w|_1 \cdot |w|_2, \dots, |w|_{k-1} \cdot |w|_k)$.

Proof: By the prior proposition we have that $\psi_2(w) + \psi_2(\tilde{w}) =$

$$(|w|_1 \cdot |w|_2, |w|_2 \cdot |w|_3, \dots, |w|_{k-1} \cdot |w|_k). \text{ Since } w \text{ is a palindrome we have that } w = \tilde{w} \Rightarrow$$

$$\psi_2(w) + \psi_2(w) = 2 \psi_2(w) = (|w|_1 \cdot |w|_2, |w|_2 \cdot |w|_3, \dots, |w|_{k-1} \cdot |w|_k) \Leftrightarrow \psi_2(w) = \frac{1}{2} (|w|_1 \cdot |w|_2, \dots, |w|_{k-1} \cdot |w|_k)$$

□

One may ask whether there are other words $w \in S_n(M)$ that are non palindromes contributes to the parikh statistic of second order. Find this out and state a lemma...

That is, is it possible for w to be such that $w \neq \overleftarrow{w}$ but $\psi_2(w) = \frac{1}{2} (|w|_1 \cdot |w|_2, \dots, |w|_{k-1} \cdot |w|_k)$?

This is equivalent with the statement $\psi_2(w) = \psi_2(\overleftarrow{w}) \Rightarrow w = \overleftarrow{w}$. But this means ψ_2 would be injective. So maybe this is possible only when ψ_2 is injective. **Another lemma?**

Question

Let v be a second order parikh subvector of the maximal parikh vector

$v \leq (|w|_1 |w|_2, \dots, |w|_{k-1} |w|_k)$ and let $\mathcal{P}_2(v) = \#\{w \in S_n(M) : \psi_2(w) \subseteq v\}$

Find a formula for $\mathcal{P}_2(v)$. Investigate its relation to $\alpha(S) = \#\{w \in S_n(M) : D(w) \subseteq S\}$. Modify $\{w \in S_n(M) : D(w) \subseteq S\}$ to find the connection between $\alpha(S)$ and $\mathcal{P}_2(v)$ in such a way that S corresponds to a vector v .

Lets view the descent set of permutations as a mapping $D : S_k \rightarrow P([k-1])$ defined by

$D(\pi) = \{i : \pi_i > \pi_{i+1}\}$, $\pi \in S_k$, $1 \leq i \leq k-1$. If we consider the *alternating descent set* S ,

defined as alternating if $\phi^{-1}(S) = v$ is alternating this corresponding to the sets

$\{2, 4, 6, \dots\} \cap [k-1]$ and $\{1, 3, 5, \dots\} \cap [k-1]$ we see that S is alternating $\Leftrightarrow v$ is alternating.

This is however not the case for the parikh mapping of second order.

Lemma: If we consider the permutation $\pi \in S_k$ that can be factored $\pi = \gamma \lambda$ such that $\gamma \in S_{\lfloor k/2 \rfloor}(\{2, 4, 6, \dots\} \cap [k-1])$. Then $\psi_2(\pi) = v_1 \cdots v_{k-1}$ is alternating such that $v_{2i} = 0$ and $v_{2i+1} = 1$, $1 \leq i \leq k-1$. There are $\lfloor k/2 \rfloor! \cdot \lceil k/2 \rceil!$ such permutations $\pi = \gamma \lambda$.

Proof : If $\gamma \in S_{\lfloor k/2 \rfloor}(\{2, 4, 6, \dots\} \cap [k-1])$ then $\lambda \in S_{\lceil k/2 \rceil}(\{1, 3, 5, \dots\} \cap [k-1])$. Since all even numbers occur to the left of all odd numbers we get that $12, 34, 56, \dots, (2i-1)2i, \dots$ does not exist as scattered subwords of π , which means that $|\pi|_{1,2} = |\pi|_{3,4} = |\pi|_{5,6} = \dots = |\pi|_{2i-1,2i} = \dots = 0$.

For the same reason we get that $23, 45, 67, \dots, (2i)2i+1, \dots$ does exist as scattered subwords in π and therefor $|\pi|_{23} = |\pi|_{45} = |\pi|_{67} = \dots = |\pi|_{2i,2i+1} = \dots = 1$. We have that $\psi_2(\pi) = v_1 \cdots v_{k-1}$ with

$v_{2i-1} = |\pi|_{2i-1,2i}$, $v_{2i} = |\pi|_{2i,2i+1}$. If k is odd, then γ is a permutation of

$\#\{2, 4, 6, \dots\} \cap [k-1] = (k-1)/2 = \lfloor k/2 \rfloor$ elements. λ is a permutation of

$k - \frac{k-1}{2} = \frac{2k - (k-1)}{2} = \frac{k+1}{2} = \lceil k/2 \rceil$ elements. If k is even we have $\#\{2, 4, 6, \dots\} \cap [k-1] =$

$\#\{1, 3, 5, \dots\} \cap [k-1] = k/2 = \lfloor k/2 \rfloor = \lceil k/2 \rceil$ elements. Since $\pi = \gamma \lambda$ it follows that there are $\lfloor k/2 \rfloor! \cdot \lceil k/2 \rceil!$ such permutations π .

Observe that these are not all permutations with alternating parikh vector. Let

$\phi(v) = \{s_1, \dots, s_l\}_<$, then by the Yamba hypothesis $\rho(v) = k! \det[1/(s_{j+1} - s_i)!]$,

$\det[1/(s_{j+1} - s_i)!]$ is a $(l+1) \times (l+1)$ determinant with (i, j) -entry

$f(i, j+1), (i, j) \in [0, k] \times [0, k]$ and $f(i, j) = 1/(s_j - s_i)!, s_0 = 0$ and $s_{l+1} = k$.

$i > j \Rightarrow f(i, j) = 0$.

That $\pi = \lambda$ is not all permutaitons with the vector $\psi_2(\pi)$ can be proved by computing the difference of the alternating descent set

$$\rho(v) = k! \begin{vmatrix} \frac{1}{(s_1 - s_0)!} & \frac{1}{(s_2 - s_0)!} & \dots & \frac{1}{(s_{l+1} - s_0)!} \\ \frac{1}{(s_1 - s_1)!} & \frac{1}{(s_2 - s_1)!} & \dots & \frac{1}{(s_{l+1} - s_1)!} \\ \dots & \dots & \frac{1}{(s_{j+1} - s_i)!} & \dots \\ \frac{1}{(s_1 - s_l)!} & \frac{1}{(s_2 - s_l)!} & \dots & \frac{1}{(s_{l+1} - s_l)!} \end{vmatrix} - \lfloor k/2 \rfloor! \lceil k/2 \rceil! > 0$$

We have $s_0 = 0, s_{l+1} = k$, and $S = \{s_1, \dots, s_l\}_< = \{2, 4, 6, \dots\} \cap [k-1]$. Note that $s_l = \lfloor k/2 \rfloor \cdot 2$, thus we have to show.

$$\rho(v) - \lfloor k/2 \rfloor! \lceil k/2 \rceil! =$$

$$k! \begin{vmatrix} \frac{1}{(2)!} & \frac{1}{(4)!} & \dots & \frac{1}{(k)!} \\ \frac{1}{(0)!} & \frac{1}{(2)!} & \dots & \frac{1}{(k-2)!} \\ \dots & \dots & \frac{1}{(2 \cdot (j+1) - 2i)!} & \dots \\ \frac{1}{(2 - \lfloor k/2 \rfloor \cdot 2)!} & \frac{1}{(4 - \lfloor k/2 \rfloor \cdot 2)!} & \dots & \frac{1}{(k - \lfloor k/2 \rfloor \cdot 2)!} \end{vmatrix} - \lfloor k/2 \rfloor! \lceil k/2 \rceil! > 0$$

$\rho(v) - \lfloor k/2 \rfloor! \lceil k/2 \rceil!$ is the number of permutations $\tau \neq \pi = \lambda \gamma$ with parikh vector

$$v = \psi_2(\tau) = \psi_2(\pi).$$

Musical Balance Theory

Defintion: A quantifiable property of a concept

Given that we have an inclusive definition of music as something contained within “*Sound over Time*” we may look deeper into what properties music has. There are properties in music that can be “quantified”. Musical elements of some sort can be assigned some a “value”, with regard to a property. By this value they can be placed in some order in relation to each other. We may forexample speak in terms of musical intensity or lack of thereof.

High pitches or low pitches. Fast tempo or slow tempo etc. So this intuitively leads us to defining musical balance as somewhere inbetween the extremes of such properties.

Definition: *partial musical balance* is the sum of values of musical elements with regard to more than one but not all musical properties of a *piece of music*.

Definition: *universal musical balance* is the sum of all values of musical elements with regard to all musical properties of a piece of music.

Definition: *specified musical balance* is the sum values of musical elements with regard to exactly one musical property.

In some sense this reminds of some sort of weight concept. Take foreexample a melody. The specified musical balance of this melody, where the specified property is the pitches of the melody. Then the balance is the sum of all pitches in the melody. If it sums up to 0 we have a “weightless” melody. Weightlessness in physics is the state where gravitational forces (or forces in general influencing an object) sums up to 0. (nonacceleration, velocity is constant) We may think of this as musical equilibrium in the same way.

Definition: *Musical equilibrium* (specified/partial or universal) is when the musical balance In question (*specified/partial or universal*) sums up to 0.

Higher levels of musical equilibriums.

In actuality it seems impossible to find the universal musical equilibrium since we may “invent”/find an uncountable number of musical properties where elements have values. The properties by which we assign a value may vary in abstractness level. We may foreexample Study how a melody moves with regard to the property of descent or what parikh matrix a melody has. Then we are on a “higher level” of musical property where we do not simply study elements in $\mathbb{R}$. We study matrices or vectors of various kind. We remind ourselves that musical balance intuitively looked like “weight”. We are weighing each element in a set, so we may ask ourself what is the weight of a parikh matrix and the idea can be generalized to all “musical elements” or maps of musical elements (=higher level properties) (parikh matrix is a map of melodies, notes(as musical elemnts.)/melodies (as musical elements))(value depends on choice of musical mapping)

Set’s as elements. Sets where elements are sets. Higher levels of musical properties.

Explore the consequences of the following properties given the above definitions

Definition. (2:nd attempt) *Musical balance* is the value of a *weight function* on a musical element of a musical set.

Definition: (Theorem) (3:rd attempt) Let M be a musical set/multiset such that M is a Lattice. (Define). Then we can define a weight function $\omega: M \rightarrow B_M \subseteq \mathbb{R}$

Thought: Instead of trying to define it generally I will have a chapter about musical balance and go through various kinds. It is a topic not one single definition or theorem.

With specifysics it is possible to define it more convincingly. If defined generally we should not attempt a mathematical definition

The musical balance function shows the distribution of weight values for musical elements.

property.1: Pitch

Definition: (musical pitch balance)

(Given that we are working in the key C. We may define the *tonal centre* to be $c_0 \leftrightarrow 0$.

We may shift the infinite key mapping in relation to the key we are working in)

The *weight* of a pitch is simply the “infinite piano” bijection priorly defined. $\omega: P \rightarrow \mathbb{Z}$ where $\omega(\phi(x)) = \omega(y) = y$. Let $M = \{x_1, \dots, x_n\} \subset P$ be a pitch set or a **pitch multiset**.

The *pitch balance* of the pitch set M is a function $b_\omega: P(P) \rightarrow \mathbb{Z}$ defined by

$$b_\omega(M) = \sum_{i=1}^n \omega(x_i). \text{ Where } P(P) \text{ is the power set of the set of all pitches.}$$

Definition: The pitch set M is said to be in *pitch equilibrium* if $b_\omega(M) = \sum_{i=1}^n \omega(x_i) = 0$.

In other words the M is a pitch equilibrium if $M \in \ker(b_\omega)$. Thus the set of all pitch

equilibriums of b_ω is simply $\ker(b_\omega)$. We see that $\{0\} \in \ker(b_\omega)$ and

$$b_\omega(\mathbb{Z}) = \sum_{i=1}^n x_i = \sum_{-n}^n x_i \rightarrow 0, n \rightarrow \infty, \text{ so } \mathbb{Z} \in \ker(b_\omega). \text{ Thus } \mathbb{Z} \text{ is a pitch equilibrium.}$$

Definition: The (*musical*) *pitch center* (in this case *tonal center* since the set is P) of P is defined as the elements of P mapping to 0. In other words the musical center of M is $x = 0$. (The middle of P $\leftrightarrow$ the middle of the set of integers)

property.2 Pitchclass is a musical property of a “higher level”. That is, a musical set that has undergone a mapping (in this case an equivalence relation on the set of pitches. $P \xrightarrow{\phi} P/\sim$). Since it is still a musical set, but on a more music-theoretical level than actual pitches.

Observe that

$P/\phi = \{0,1,2,3,4,5,6,7,8,9,10,11\}$, so if we adopt the definitions for P, the weight $\omega(x)$ is the same as for pitches and

$$b_\omega(M_\sim): P(P/\sim) \rightarrow \mathbb{N} \text{ defined by } b_\omega(M_\sim) = \sum_{x \in M_\sim \subset P/\sim} \omega(x) = 0 \Leftrightarrow x = 0, \forall x \in M_\sim.$$

Therefor I will modify the definitions abit, since we want to find pitchclass sets/multisets that are musical equilibriums other than the trivial M containing only occurrences of 0.

Definition: The pitch class center C of $P/\sim$ of order 1 is defined as the set

$$C = \left\{ \left\lfloor \frac{\min(P/\sim) + \max(P/\sim)}{2} \right\rfloor, \left\lceil \frac{\min(P/\sim) + \max(P/\sim)}{2} \right\rceil \right\} = \{5, 6\}. \text{ The } \textit{pitchclass center} \text{ of}$$

order k is defined as the set

$$C^k = \left\{ \left\lfloor \frac{k \cdot \min(P/\sim) + k \cdot \max(P/\sim)}{2} \right\rfloor, \left\lceil \frac{k \cdot \min(P/\sim) + k \cdot \max(P/\sim)}{2} \right\rceil \right\} = \{k \cdot 5, k \cdot 6\}$$

I was aiming for a discrete definition.

Observe that C^k is a set not a multiset although we are working with those when it comes to Musical pitchclasses and there balance in general. So we have for $k=1,2,3,4,5,6,\dots$ the following centers C^k $\{5,6\}, \{11\}, \{16,17\}, \{33\}, \{49,50\}, \{99\}, \dots$

Definition: Let $P(P_M / \sim)$ be the powerset of all *pitchclass multisets*. The *pitchclass balance* of a set/multiset $M_{\sim}$, where $\#M_{\sim} = k$, is a map $b_{\omega} : P(P_M / \sim) \rightarrow \mathbb{Z}$ defined by

$$b_{\omega}(M_{\sim}) = \sum_{x \in M_{\sim} \subseteq P(P_M / \sim)} \omega(x) - z, \text{ where } z \in C^k.$$

Definition Let $M_{P/\sim} \subset P(P_M / \sim)$, where $\#M_{P/\sim} = k$. The pitchclass set/multiset of $M_{P/\sim}$ is a *pitchclass equilibrium* if $b_{\omega}(M_{P/\sim}) = \sum_{x \in M_{P/\sim} \subseteq P(P_M / \sim)} \omega(x) = z$ for some $z \in C^k$ or equivalently $M_{P/\sim} \in \ker(b_{\omega})$.

Property.3: *Musical timing*. This property deals with **when** something occurs in music in relation to other things occurring, so it is about the relation between sound and . We may trace *musical events* within the music and find there is some sort of *meaning* for something to occur in the music. We may speak in terms of *cause and effect* and the order of musical sets is relevant. It is a complex property but nonetheless important in music. This is a suggestion of mathematically describing certain aspects of musical timing.

Property.4: *Rhythm*.

There is a bijection between the set of all rhythms R and $\mathbb{Q}$. $\phi : R \rightarrow \mathbb{Q}$, $x \mapsto \frac{p}{q}$.

By convention music is considered having an underlying time signature, $\frac{p}{q} \in \mathbb{Q}$, where p is

the number of *beats* $\frac{1}{q}$ per bar (*relative duration*) and where the values for q is, by

convention, almost exclusively $q = 2^k$. Also, by convention, the duration is set for the music as the number of $\frac{x}{q} = 60$ (60 seconds). Thus the number of *beats* $\frac{1}{q}$ per minute (*physical duration*).

Definition Let $M_R \subset P(R_M)$ be a set/multiset of rhythms and $P(R_M)$ be the set of all multisets of rhythms. The k :th *rhythmic center* C^k , $k \geq 1$, of rhythmic set/multiset $M_R \subset P(R_M)$, with $\#M_R = k$ is defined as $C^k = \frac{k}{q} \in \mathbb{Q}^+$

Definition Let $M_R \subset P(R_M)$ be a set/multiset of rhythms and $P(R_M)$ be the set of all multisets of rhythms. The *rhythm weight* is a map $\omega : P(R_M) \rightarrow \mathbb{Q}^+$ defined by $\omega(M_R) = \sum_{x \in M_R} x$.

Definition Let $M_R \subset P(R_M)$ be a set/multiset of rhythms and $P(R_M)$ be the set of all multisets of rhythms. The *rhythm balance* of $M_R \subset P(R_M)$, $\#M_R = k$, is a map $b_{\omega} : P(R_M) \rightarrow \mathbb{Z}$ defined by $b_{\omega}(M_R) = \log_q(\omega(M_R) - C^k)$.

Definition: A set/multiset of rhythms, $M_R \subset P(R_M)$, $\#M_R = k$, is a *rhythmic equilibrium* if $b_{\omega}(M_R) = \log_q(\omega(M_R) - C^k) = 0$. That is, M_R is a rhythmic equilibrium if $c \in \ker(b_{\omega}(M_R))$

Melodical Properties: (*pitch and pitchclass and various types of permutation sets like descent sets, excedance parikh sets etc*)

Property.5: *Descent set*

The descent set of a melodic vector (permutation of a multiset) can be defined by a map $D: S_n(M) \rightarrow [n-1]$ such that $D(w) = \{i: w_i > w_{i+1}\}$, $w \in S_n(M)$.

Definition The *descent weight* of a melodic vector is defined as $d(w) = \#D(w)$.

Definition The *descent center* is defined as a set (not a multiset)

$$C(S_n(M)) = \left(\lfloor d(w)/2 \rfloor, \lceil d(w)/2 \rceil \right), w \in S_n(M).$$

Definition The *descent balance* of a melodic vector is a function $b_{d(w)}: S_n(M) \rightarrow \mathbb{Z}$, defined by $b_{d(w)}(w) = d(w) - z$, where $z \in C(S_n(M))$.

Definition The melodic vector w is a *descent equilibrium* if $b_{d(w)}(w) = d(w) - z = 0$, where $z \in C(S_n(M))$, or equivalently, the melodic vector w is a *descent equilibrium* if $w \in \ker(b_{d(w)})$.

Property.6: Parikh set

$$M = \{1^{a_1}, \dots, k^{a_k}\}, \#M = \sum_{i=1}^k a_i = n.$$

Definition The parikh set of a melodic vector w can be defined by the map

$$\psi_2: S_n(M) \rightarrow \mathbb{N}^{k-1} \text{ by } \psi_2(w) = (|w|_{12}, \dots, |w|_{(k-1)k}).$$

Definition The *parikh weight* is the value of the function $\omega_{\psi_2}: S_n(M) \rightarrow \mathbb{N}$ defined by

$$\omega_{\psi_2}(w) = |w|_{12} + \dots + |w|_{(k-1)k}.$$

Definition The *parikh center* is defined as the set (not multiset)

$$C_{\psi_2}(S_n(M)) = \left(\lfloor n/2 \rfloor, \lceil n/2 \rceil \right)$$

Definition The *parikh balance* is the value of the function $b_{\omega_{\psi_2}}(w) = \omega_{\psi_2}(w) - z$, $z \in C_{\psi_2}(S_n(M))$.

Definition A melodic vector w is a *parikh equilibrium* if $b_{\omega_{\psi_2}}(w) = \omega_{\psi_2}(w) - z = 0$, $z \in C_{\psi_2}(S_n(M))$, or equivalently a melodic vector w , is a *parikh equilibrium* if $w \in \ker(b_{\omega_{\psi_2}})$.

Property excedance set

Definition The *excedance set* can be defined by the map $E: S_n(M) \rightarrow \mathbb{N}^{n-1}$ where

$$E(w) = \{i: w_i > i\}.$$

Definition Let $w \in S_n(M)$

The *excedance center* is defined as the set (not multiset) $C(S_n(M)) = \left(\lfloor n/2 \rfloor, \lceil n/2 \rceil \right)$.

Definition The *excedance weight* is a the value of a function $\omega_E: S_n(M) \rightarrow \mathbb{N}$ defined by $\omega_E(w) = e(w) = \#E(w)$, where $w \in S_n(M)$

Definition The *excedance balance* of w is a function $b_e: S_n(M) \rightarrow \mathbb{Z}$ defined by $b_e(w) = e(w) - z$, for some $z \in C_e(S_n(M))$.

Definition The a melodic vector $w \in S_n(M)$ is a *exedance equilibrium* if $b_e(w) = e(w) = z$, or equivalently, the a melodic vector $w \in S_n(M)$ is a *exedance equilibrium* if $w \in \ker(b_{\omega_e})$.

(Property.7: Tempo “hypothesis” - the tempo center is the average number of beats per minute of a human heart)

Definition The *tempo* of a piece of music c is the number of $\frac{1}{q}$ - beats per minute.

Definition Let the *tempo center* be the average number of heartbeats per minute and denote this by H .

Definition Let p denote the number of $\frac{1}{q}$ - beats and let t denote time. The *tempo weight* of a time interval of a piece of music c is defined as $\omega(c) = \frac{p}{t}$, where.

Definition: Let p denote the number of $\frac{1}{q}$ - beats and let t denote time. The *tempo balance* of an interval of time of a piece of music, denoted by c , is the value of the function $b_\omega : T \rightarrow \mathbb{Z}$ defined by $b_\omega(c) = \omega(c) - H$.

Definition: An interval of a time of a piece of music, denoted by c , is a *tempo equilibrium* if $\omega(c) = H$ or equivalently $b_\omega(c) = \omega(c) - H = 0$. That is c is a tempo equilibrium if $c \in \ker(b_\omega(c))$.

Property.8: Pausing/articulation

If we take an interval of time in a piece of music that contains no musical elements we are talking about what is intuitively recognized as silence, or if u like we could call it empty sound (No or close to no “acoustic vibrations” (sound waves (straight line))). If we are interested in how these silence occur in relation to the sound in a piece of music, over time. we are studying *the articulation of pausing*.

Definition Let the $\frac{r}{s} \in \mathbb{Q}$, ($q = 2^k$ for all rhythms $\frac{p}{q}$ of c) be the rhythm in a piece of music c , with minimal relative duration.

The *atom-beat* a , of a piece of music c , is defined as $a = c / r = \frac{1}{s}$.

The number of atom-beats of a rhythm $e = \frac{p}{q} \in \mathbb{Q}^+$ is defined as $a(e) = s \cdot e$.

The number of atom-beats of a pause is defined eqiovalently.

Definition Let M_e denote the the set/multiset of pauses and M_R be denote the set/multiset of rhythms of a piece of music c . Let the *total number of atom-beats* of c be denoted by

$$A(e, r) = \sum_{e \in M_e \cup r \in M_R} (a(e) + a(r))$$

Definition: *pause articulation weight* of an interval of time of a piece of music, c , is a function

$$\omega_{p.A.} : S(T) \rightarrow \mathbb{Z}^+ \text{ defined by } \omega_{p.A.}(c) = \sum_{r \in M_R} a(r), c \in S(T).$$

$S(T) \leftrightarrow$ Sound over time

Definition: The *pause-articulation center* is defined as the set (not multiset)

$$C(c) = \{ \lfloor \omega_{p.A.}(c) / 2 \rfloor, \lceil \omega_{p.A.}(c) / 2 \rceil \}.$$

Definition The *pause-articulation balance* of a piece of music c , is the value of a function

$$b_{\omega_{p.A.}} : S(T) \rightarrow \mathbb{Z}$$

defined by $b_{\omega_{p,A.}}(c) = \sum_{e \in M_e} a(e) - C(c)$.

Definition A piece of music c is a *pause-articulation equilibrium* if

$b_{\omega_{p,A.}}(c) = \sum_{e \in M_e} a(e) - C(c) = 0$. Equivalently a piece of music c , is a pause-articulation equilibrium if $c \in \ker(b_{\omega_{p,A.}})$.

Musical balance is a difficult topic. When considered by definitions above, one composition, might be “unbalanced” when it is considered by itself, but it might be intended as part of a whole suite of compositions in such a way that they compensate each other, resulting in a combined balanced closer to musical equilibrium.

Music-theoretical Hypothesis: If c_1 and c_2 are *musical compositions* such that

$b_{\omega}(c_1) < b_{\omega}(c_2)$ for all *musical properties* that are *quantifiable*.

Regarding Evidence: This is not proofable in strict terms. We could aim for finding evidence in support of the hypothesis and expect criticism and counter evidence to some degree. But my guess is that it will be more evidence in favour of the hypothesis rather than against so it cannot be dismissed. Criticism will be healthy and fruitful for modifying and further developing the theory of musical balance.

Silence is empty sound...

q-expansions of various kinds with music theoretical applications

The use of q-expansion is very effective in providing more enumerative information of whatever we are examining.

Given a melodic vector of n -tone series. Let's say we want construct another melodic vector such that the vector difference vector between the new melody and its original has no entry less than zero. Then we are dealing with, what in mathematical terms, is known as inversion of a permutation.

Definition: A pair (π_i, π_j) is called an inversion of $\pi = \pi_1 \cdots \pi_n$ if $i < j$ and $\pi_i > \pi_j$

The following definition is one way to represent a permutation with regard to all its inversions

Definition: the vector $I(\pi) = (a_1, \dots, a_n)$ where $0 \leq a_i \leq n - i$, where a_i is the number of entries j of $\pi \in S_n$ that are to the left of i , such that $j > i$, is called the *inversion table* of π . Let's denote the set of all inversion tables of π by $\mathfrak{I}_n = \{ I(\pi) = (a_1, \dots, a_n) : 0 \leq a_i \leq n - i, \pi \in S_n \}$.

Since $0 \leq a_i \leq n - i, 1 \leq i \leq n$ we can also represent the set of inversion tables as

$\mathfrak{I}_n = [0, n - 1] \times [0, n - 2] \times \cdots \times [0, 0]$.

Proposition: $I : S_n \rightarrow \mathfrak{I}_n$ is a bijection.

It is clear that $\# \mathfrak{I}_n = |[0, n - 1]| \times |[0, n - 2]| \times \cdots \times |[0, 0]| = n \cdot (n - 1) \cdots 1 = n!$,

so to show that there is a bijection between $\mathfrak{I}_n$ and S_n it suffices to show an injection or surjection. But first we will make some useful definitions:

Definition: a *partial inversion table* $I_k(\pi)$ is a vector $(a_{n-k+1}, \dots, a_n)$, where $0 \leq a_i \leq n - i$.

Let $i = \pi_m$ and $a_i = \#\{\pi_j \in \pi : \pi_j > i = \pi_m, j < m\}$.

Definition: Let $S_n^k \subseteq P_n^k$ denote the set of all partial permutations such that $\pi \in S_n^k$ is a permutation of the k elements $[n - k + 1, n]$.

Proof: (by induction) (initial conditions) (hmm try surjection later)

Lets assume that $I : S_n^k \rightarrow \mathfrak{S}_k$ is a bijection and observe that we have $I : S_n^k \rightarrow \mathfrak{S}_k$ with $k = n$
 $\Leftrightarrow I : S_n \rightarrow \mathfrak{S}_n$.

Take an arbitrary $\pi = \pi_{n-k+1} \cdots \pi_n \in S_n^k$. $I(\pi) = (a_{n-k+1}, \dots, a_n)$, $0 \leq a_i \leq n - i$. The least element in π is $\pi_{a_{n-k+1}}$. Take a look at $I : S_n^{k+1} \rightarrow \mathfrak{S}_{k+1}$. Each $\lambda \in S_n^{k+1}$, consisting of the permutations of elements $[n - k + 1, n]$, can be achieved by taking a $\pi = \pi_{n-k+1} \cdots \pi_n \in S_n^{k+1}$, and assigning $\lambda_l = n - k$, $\lambda_j = \pi_j$ if $j > l, 1 \leq j \leq k + 1$. This assigning can be done in $(k + 1)!$ ways

For each of the $k!$ permutations of S_n^k , resulting in, as we desired, the $(k + 1)!$ unique partial permutations S_n^{k+1} . We observe that $\#\mathfrak{S}_{n+1} = (n + 1)!$. If $I(\pi) = (a_{n-k+1}, \dots, a_n)$, $0 \leq a_i \leq n - i$.

Then the construction of $\lambda \in S_n^{k+1}$ by assigning $\lambda_l = n - k$ etc, will result in

$I(\lambda) = (a_{n-k}, a_{n-k+1}, \dots, a_n)$, $0 \leq a_i \leq n - i$, since $\lambda_l = n - k$, being less than all elements in π , means that $I(\pi) = (a_{n-k+1}, \dots, a_n)$, $0 \leq a_i \leq n - i$, goes unchanged through $I(\lambda)$.

Now take $I(\lambda) = I(\gamma)$, $\lambda, \gamma \in S_n^k$. We have that $I(\lambda) = (a_{n-k}, \dots, a_n) = (b_{n-k}, \dots, b_n) = I(\gamma)$.

Since $I(\pi) = (a_{n-k}, \dots, a_n)$ and $a_{n-k} = b_{n-k}$ corresponds to $\lambda_{l=a_{n-k}} = \lambda_{b_{n-k}} = n - k$, we have that

$\lambda_{l=a_{n-k}} = n - k = \gamma_{b_{n-k}}$. This together with our assumption that $I : S_n^k \rightarrow \mathfrak{S}_k$ is a bijection leads us to $I(\lambda) = I(\gamma) \Rightarrow \lambda = \gamma$, injectivity. Thus $I : S_n^{k+1} \rightarrow \mathfrak{S}_{k+1}$ is a bijection since $\#\mathfrak{S}_n^{k+1} = \#S_{n+1}$

$I : S_n^{k+1} \rightarrow \mathfrak{S}_{k+1}$ is a bijection. $\square$

Definition: Let $i(\pi)$ denote the permutation statistic $i(\pi)$

$\#\{(\pi_i, \pi_j) : \pi = \pi_1 \cdots \pi_n \in S_n, 1 \leq i < j \leq n, \pi_i > \pi_j\}$, in other words $i(\pi)$ is the number of inversions of $\pi \in S_n$.

Corollary: $\sum_{\pi \in S_n} q^{i(\pi)} = (1)(1 + q)(1 + q + q^2) \cdots (1 + q + q^2 + \cdots + q^{n-1})$.

Proof: By the definition of $I(\pi)$ and $i(\pi)$, $i(\pi) = a_1 + \cdots + a_n$. Thus

$$\sum_{\pi \in S_n} q^{i(\pi)} = \sum_{a_1=0}^{n-1} \sum_{a_2=0}^{n-2} \cdots \sum_{a_n=0}^0 q^{a_1 + \cdots + a_n} = \sum_{a_1=0}^{n-1} q^{a_1} \sum_{a_2=0}^{n-2} q^{a_2} \cdots \sum_{a_n=0}^{n-1} q^{a_n} \quad \square$$

We have mostly studied the special case of melodies, namely n -tone series, practised in “serial music” forexample 12-tone music. However in music in general, melodies contain repetition of notes and thus we are looking at permutations of multisets. We may ask ourselves, how does various enumerative properties of n -tone series like permutatation-statistics, like $i(\pi)$, translate to melodies where repetition is allowed, if we are looking at $\pi \in M_n$, where $M_n = \{1^{a_1}, \dots, m^{a_m}\}$?

Take the melodic vector of the first phrase of Jimi Hendrix “voodoo child”

$(e_1, d_1, h_0, d_1, g_0, g_0, a_0, g_0, e_0, e_1, d_1, h_0, e_1)$ correponds to $(4, 2, -1, 2, -5, -5, -3, -5, -9, 4, 2, -1, 4)$ if pitches are translated into correspondng elements of $\mathbb{Z}$. The melodic vector can be seen as a permutation of $M = \{-9, -5, -5, -3, -1, -1, 2, 2, 2, 4, 4, 4\}$, more conveniently written as

$M = \{-9, (-5)^2, -3, (-1)^2, 2^3, 4^3\} = \{1, 2^2, 3, 4^2, 5^3, 6^3\}$. We get

$(4, 2, -1, 2, -5, -5, -3, -5, -9, 4, 2, -1, 4) = (1, 2, 3, 2, 5, 5, 4, 5, 6, 1, 2, 3, 1)$.

Now let's make a definition that will give oversight when enumerating permutations of multisets.

Definition: Let $M = \{1^{a_1}, \dots, m^{a_m}\}$ be a multiset with $\#M = a_1 + \dots + a_m$. Define a *multiset permutation vector* as a vector $(\pi_0, \pi_1, \dots, \pi_m)$, where $\pi_0 \in S_n(M)$, $\pi_i \in S_{a_i}$, $1 \leq i \leq m$.

Let's denote the set of permutation vectors $(\pi_0, \pi_1, \dots, \pi_m)$ as $S_n(M) \times S_{a_1} \times \dots \times S_{a_m}$.

Example: To demonstrate application of this definition take $M = \{1, 2^2, 3, 4^2, 5^3, 6^3\}$ and let $\pi_0 \in S_n(M)$ be the above melody $(1, 2, 3, 2, 5, 5, 4, 5, 6, 1, 2, 3, 1)$. We have

$\#M = 1 + 2 + 1 + 2 + 3 + 3 = 12$ and the permutations $\pi_i \in S_{a_i}$, $1 \leq i \leq m$ is

$\pi_1 \in S_1, \pi_2 \in S_2, \pi_3 \in S_1, \pi_4 \in S_2, \pi_5 \in S_3, \pi_6 \in S_3$ and when determining the number of permutations of length $M = \{1, 2^2, 3, 4^2, 5^3, 6^3\}$, we may first consider repeated elements as distinct by renaming them, for example 5,5,5 into $5_1, 5_2, 5_3$ etc. We then get $12!$ permutations. But since the order of repeated elements like $5_1, 5_2, 5_3$, is disregarded for $\pi_0 \in S_n(M)$ we need to divide by all $a_i!$, $1 \leq i \leq m$. This is in accordance with the multinomial theorem we have

$$\frac{|S_n|}{|S_{a_1}| \cdots |S_{a_m}|} = |S_n(M)| = \frac{n!}{a_1! \cdots a_m!} = \binom{n}{a_1, \dots, a_m}, \#M = a_1 + \dots + a_m.$$

$$\text{And so } |S_n(M) \times S_{a_1} \times \dots \times S_{a_m}| = \binom{n}{a_1, \dots, a_m} \cdot (a_1)! \cdots (a_m)! = n! = |S_n|!$$

This hints that there is a bijection between $S_n(M) \times S_{a_1} \times \dots \times S_{a_m}$ and S_n . All we need to do is to find a mapping $\phi: S_n(M) \times S_{a_1} \times \dots \times S_{a_m} \rightarrow S_n$ and show surjection or injection for ϕ since we already know $|S_n(M) \times S_{a_1} \times \dots \times S_{a_m}| = |S_n|!$.

Definition: Let $(a_1, \dots, a_m)$ be a sequence of nonnegative integers such that $\sum_{i=1}^m a_i = n$ then the

q-multinomial coefficient $\binom{n}{a_1, \dots, a_m}_q = \mathbf{n!} / ((\mathbf{a}_1)! \sqcup (\mathbf{a}_m)!)$, where $\mathbf{k}! = (1)(2) \sqcup (\mathbf{k})$ and $(\mathbf{j}) =$

$(1 + q + q^2 + \dots + q^{j-1})$. We may think of it as a rational function in q when evaluated at $q = 1$

reveals the ordinary multinomial coefficient. $\binom{n}{a_1, \dots, a_m}_1 = \binom{n}{a_1, \dots, a_m}$

We have that

$$1) \binom{n}{a_1, \dots, a_m}_q = \binom{n}{a_1}_q \binom{n-a_1}{a_2}_q \binom{n-a_1-a_2}{a_3}_q \dots \binom{a_m}{a_m}_q$$

$$2) \binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k} \binom{n-1}{k-1}_q, \text{ is known as the } q\text{-binomial coefficient.}$$

Proposition: Let $M = \{1^{a_1}, \dots, m^{a_m}\}$ be a multiset with $\#M = a_1 + \dots + a_m$. Then

$$\sum_{\pi \in S_n(M)} q^{i(\pi)} = \mathbf{n!} / ((\mathbf{a}_1)! \sqcup (\mathbf{a}_m)!)$$

Proof: Define a map $\phi: S_n(M) \times S_{a_1} \times \cdots \times S_{a_m} \rightarrow S_n$ with $(\pi_1, \pi_2, \dots, \pi_m) \rightarrow \pi$.

by letting $a_0 = 0$ and $1^{a_1} = (\underbrace{1 \ 1 \ \cdots \ 1}_{a_1})$ and converting the a_i i's in multi-permutation π_0

$$(1^{a_1}) \rightarrow (1, 2, \dots, a_1)$$

$$(2^{a_2}) \rightarrow (a_1 + 1, a_1 + 2, \dots, a_1 + a_2)$$

$$(3^{a_3}) \rightarrow (a_1 + a_2 + 1, a_1 + a_2 + 2, \dots, a_1 + a_2 + a_3)$$

.....

$$(m^{a_m}) \rightarrow (a_1 + a_2 + \cdots + a_{m-1} + 1, a_1 + a_2 + \cdots + a_{m-1} + 2, \dots, a_1 + a_2 + \cdots + a_{m-1} + a_m)$$

By reversing this we see that the map is an bijection because given $(a_1, \dots, a_m)$ with $\sum_{i=1}^m a_i = n$

any permutation in S_n can be decomposed into a multi-permutation in $S_n(M)$.

We have that $i(\pi) = i(\pi_0) + i(\pi_1) + \dots + i(\pi_m)$ (show)

By corollary it follows that $(\sum_{\pi \in S_n} q^{i(\pi)}) (\mathbf{a}_1)! \cdots (\mathbf{a}_m)! = (\mathbf{n})!$ $\square$

Partitions of n and enumeration of special types of melodies.

Lets put ourselves in the shoes of a musical improviser and ask ourselves. Given two notes a and b, lets say I want create a melody $(x_1 = a, x_2, \dots, x_{n-1}, x_n = b)$ such that all notes in the melody is rising (all intervals between consecutive notes are nonnegative) and

$Int(a, b) = b - a = n > 0$. Then what we're working with is essentially the directed intervalk

vector $\langle x_2 - x_1, x_3 - x_2, \dots, x_{k-2} - x_{k-3} \rangle$, where $x_i - x_{i-1} \geq 0$. If we let $x_i - x_{i-1} = \lambda_i \in \mathbb{N}$,

we can denote this by a sequence $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$. We have $Int(a, b) = b - a = n > 0$

and thus $\lambda_1 + \cdots + \lambda_k = n$. The order of the intervals λ_i is relevant.

Definition: A composition of n into k parts, a k-composition, is a sequence of positive integers denoted by $c_k(n) = (\lambda_1, \dots, \lambda_k)$, where $\lambda_1 + \cdots + \lambda_k = n$.

Example: Lets say we are to create a melody from $c_1 = 0$ to $e_4 = 4$. $Int(c_1, e_1) = 4 - 0 = 4$.

There are eight composition of this interval with corresponding melodies, represent these of the form. Let $\phi: P \rightarrow \mathbb{Z}$, where P is the set of pitches.

$$\lambda_1 + \cdots + \lambda_k \rightarrow (\lambda_1, \dots, \lambda_k) \rightarrow (0, 0 + \lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \cdots + \lambda_k) \rightarrow (\phi^{-1}(0), \phi^{-1}(\lambda_1), \phi^{-1}(\lambda_1 + \lambda_2), \dots, \phi^{-1}(\lambda_1 + \cdots + \lambda_k)) \\ \rightarrow \{\lambda_1, \dots, \lambda_1 + \cdots + \lambda_{k-1}\} \subseteq [n - 1]$$

$$1+1+1+1 \rightarrow (1, 1, 1, 1) \rightarrow (0, 1, 2, 3, 4) \rightarrow (c_1, d_1, e_{b1}, e_1) \rightarrow \{1, 2, 3\}$$

$$2+1+1 \rightarrow (2, 1, 1) \rightarrow (0, 2, 3, 4) \rightarrow (c_1, d_1, e_{b1}, e_1) \rightarrow \{2, 3\}$$

$$1+2+1 \rightarrow (1, 2, 1) \rightarrow (0, 1, 3, 4) \rightarrow (c_1, c_{\#1}, e_{b1}, e_1) \rightarrow \{1, 3\}$$

$$1+1+2 \rightarrow (1, 1, 2) \rightarrow (0, 1, 2, 4) \rightarrow (c_1, d_{b1}, d_1, e_1) \rightarrow \{1, 2\}$$

$$3+1 \rightarrow (3, 1) \rightarrow (0, 3, 4) \rightarrow (c_1, e_{1b}, e_1) \rightarrow \{3\}$$

$$1+3 \rightarrow (1, 3) \rightarrow (0, 1, 4) \rightarrow (c_1, c_{\#1}, e_1) \rightarrow \{1\}$$

$$2+2 \rightarrow (2, 2) \rightarrow (0, 2, 4) \rightarrow (c_1, d_1, e_1) \rightarrow \{2\}$$

$$4 \rightarrow (4) \rightarrow (0, 4) \rightarrow (c_1, e_4) \rightarrow \{\emptyset\}$$

Such a melody is ofcourse completely determined by the intervals chosen between the first pitch 0 and the last $(\lambda_1 + \cdots + \lambda_k)$ and it is easy to that chosing these intervals

$(\lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \cdots + \lambda_{k-1})$ corresponds to choosing a subset of $[1, k - 1]$.

Thus each composition is determined uniquely by $(\lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \cdots + \lambda_k)$.

We have $\sum_{k=1}^{n-1} \binom{n-1}{k-1} = 2^{n-1}$ of melodies of this type given $Int(a, b) = n$.

Proposition: Let $C(n)$ denote the set of all compositions of $n \in \mathbb{Z}^+$ and $P([n-1])$ be the powerset of $[n-1]$ then the map

$\theta: C_k(n) \rightarrow P([n-1])$ is a bijection. $\square$

We could easily dwell deeper into the theory of compositions of n but since we are interested in certain q -expansions, so we will instead continue with partitions of n .

If we do not distinguish the order of the summands of an integer, that is if we consider λ and $\mathcal{Y}$ equal when the length of λ equals $\mathcal{Y}$ we are dealing with a *partition* of n . We may choose one order, largest element first and decreasing to represent equal compositions.

Definition: A *partition* of $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$, where $\lambda_1 + \dots + \lambda_k = n$ and $\lambda_1 \geq \dots \geq \lambda_k$. We regard two sequences λ and $\mathcal{Y}$ as equal if they only differ by zeros.

Forexample $\lambda = (2, 7, 3, 0, 0, 1) = (2, 7, 3, 1) = \mathcal{Y}$. In the future we will consider only

$\lambda = (\lambda_1, \dots, \lambda_k)$, where $\lambda_k > 0$. $\lambda \preceq \lambda_1, \dots, \lambda_k >$. If partition has a_i parts equal to i , we have

$\lambda \preceq 1^{a_1}, 2^{a_2}, \dots, m^{a_m} >$. We have three definitions of the partition cases

- 1) $p(n) :=$ the total number of partitions of n
- 2) $p_k(n) :=$ the number of partitions with k parts
- 3) $p(n, k, j) :=$ the number of partitions of n into k parts with largest part $\leq j$.

Example:

Proposition: Let $p_0(0) = p(0) = 1$. $p_k(n)$ satisfies the recurrence

$$p_k(n) = p_{k-1}(n-1) + p_k(n-k).$$

Proof

A partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n may be presented as an array of n dots such that the number of dots of row i is λ_i . Such a diagram is called a *Ferrer diagram*. If we replace each dot by a square we have another equivalent representation of the same partition called a *Young diagram*. Given the Ferrer diagram of one partition of n we may construct another by interchanging the rows and columns. In other words if we have the directed interval vector of a melody from a to b we can of creating a new melody between a and b by using the Ferrer diagram or the corresponding definition

Definition: The conjugate of $\lambda = (\lambda_1, \dots, \lambda_k)$ is $\bar{\lambda} = (\lambda'_1, \dots, \lambda'_m)$ where $\bar{\lambda}_1 = k$ and $m = \lambda_1$.

The number of parts in $\bar{\lambda}$ that equal i is in $\lambda_i - \lambda_{i+1}$.

We may write this procedure $\bar{\lambda} = (\lambda_1 - \lambda_2, \dots, \lambda_k - \lambda_{k+1}) = (\lambda'_1, \dots, \lambda'_m)$, where we assume $\lambda_{k+1} = 0$. The $k - m$ 0's in $\bar{\lambda}$ are omitted.

Comment It is a little the curious that the directed interval set of the directed interval set is the conjugate partition. The conjugate operation is idempotent.

Our study of the enumeration of directed intervals of melodies from a to b will now touch up on linear algebra. We shall see that the q -binomial coefficient has an interesting combinatorial interpretation concerning subspaces of a vector space. We shall also see that this we can use

this to more information about $p(j, k, n)$. $p(j, k, n)$ will appear as coefficients of the q -polynomial $\binom{j+k}{j}_q$, where n is the exponent.

(slå ihop bevis 1 och 2 ? följande e bevis 1)

Proposition: Let q denote a prime power, let F_q denote a finite field with q elements. and let $V_m(q) = F_q^m = \{(a_1, \dots, a_m) : a_i \in F_q\}$ be the m -dimensional vector space of over F_q .

Then the number of k -dimensional subspaces of $V_m(q)$ is $\binom{m}{k}_q$.

Proof: Lets denote the number of k -dimensional subspaces by $G(m, k) = \binom{m}{k}_q$.

Let $(v_1, \dots, v_k)$ be a linearly independent vector in $V_m(q)$. We may construct a $(v_1, \dots, v_k)$ by first choosing v_1 in $q^m - 1$ ways, v_2 in $q^m - q$ ways etc. or we may first choose a k -dimensional subspace $V_k(q)$ of $V_m(q)$ in $G(m, k)$ ways and then choose $v_1 \in V_k(q)$ in $q^k - 1$ ways, $v_2 \in V_k(q)$ in $q^k - q$ etc. Thus

$$(q^m - 1)(q^m - q) \cdots (q^m - q^{k-1}) = G(m, k) q^k (q^k - 1)(q^k - q) \cdots (q^k - q^{k-1}) \Rightarrow$$

$$= \frac{(q^m - 1)(q^m - q) \cdots (q^m - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})} = \binom{m}{k}_q$$

In the next proposition we will get more enumerative information on the number of melodies from a to b with a given div vector, with regard to $p(j, k, n)$. The proof will make use of some more use of linear algebra. A k -dimensional subspace of $V_m(q)$ may be expressed has a unique ordered basis $(v_1, \dots, v_k)$ that can be represented as a certain matrix. Lets define this matrix.

Definition: A matrix in *row-reduced echelon form* is a defined by: (find an alternative to stanleys)

Example:

Proposition Fix $j, k \in \mathbb{N}$. Then

$$\sum_{n \geq 0} p(j, k, n) q^n = \binom{j+k}{j}_q.$$

Proof:

Theory and Examples: the partitions of n by $p(n, j, k)$ can be represented with all Young diagrams that fit into a $k \times j$ rectangle.

Theory and examples:

Structure of selfconjugate melodies. Number of selfconjugate melodies.

Melodies, Parikh matrices and their q -expansion

followed by contemplative epilog and speculation on futher studies and possibilities.

Descentset function as a musical operation on a melodies set of intervals. Inversion of signs

$$< x_2 - x_1, x_3 - x_2, \dots, x_n - x_{n-1} >$$

Musical timing and parikh matrix encoding

Definitions

Let $\Sigma = \{a_1 < \dots < a_k\}$

let $a_{i,j} = a_i a_{i+1} \dots a_{j-1} a_j$, $a_{i,j} \in \Sigma^*$

Let $w = u_i a_i u_{i+1} \dots u_j a_j u_{j+1}$, $w \in \Sigma^*$ such that $u_l \in \Sigma^*$, $1 \leq l \leq k+1$ and $i \leq l \leq j+1$.

When we study timing we are interested in when something occurs in relation to what occurs as a whole. If we specifically are interested in when notes of a *scattered submelody* occur in relation to it's *supermelody* we are interested in the positions of the letters of

$a_{i,j} = a_i a_{i+1} \dots a_{j-1} a_j$ in $w = u_i a_i u_{i+1} \dots u_j a_j u_{j+1}$.

Let the *length* of u_l be denoted by $|u_l|$. Then the positions of $a_i, a_{i+1}, \dots, a_j$ in w are given by

$I_i = |u_i| + 1$, $I_{i+1} = |u_i| + |u_{i+1}| + 2, \dots$, $I_j = |u_i| + |u_{i+1}| + \dots + |u_j| + j - i + 1$.

We may define a weight of the timing of a submelody by considering their positions as contributing to this weight. If weight of a submelody is minimal it occurs "early" in the melody. If the weight of a submelody is maximal it should appear "late" in the melody. If the melody is a *timing equilibrium* we would wish it to occur "in the middle" of the melody. This makes it natural to define *timing weight*, *timing center*, *timing balance*, *timing equilibrium* in the following way.

Since we are working with multisets there might be several occurrences of the same word $a_{i,j}$ as a scattered word in w . Each of these occurrences have ofcourse a distinct *vector of positions* ${}^s I_{i,j} = ({}^s I_i, {}^s I_{i+1}, \dots, {}^s I_j)$. So if we wish to list these occurrences, we need to find a way to totally order the vectors ${}^s I = ({}^s I_i, {}^s I_{i+1}, \dots, {}^s I_j)$. So for a subword $a_{i,j}$ with $|w|_{a_{i,j}}$ occurrences in w , we want to have a vector $I_{i,j} = ({}^1 I_{i,j}, {}^2 I_{i,j}, \dots, {}^{|w|_{a_{i,j}}} I_{i,j})$

Occurrence specifications s of scattered subwords of a word

Definition: Let $\Sigma = \{a_1 < \dots < a_k\}$

Let $\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s$ mean the s :th occurrence of $a_{i,j}$ as a scattered subword in w , having the distinct positions ${}^s I = ({}^s I_i, {}^s I_{i+1}, \dots, {}^s I_j)$.

The *timing weight* of the s :th occurrence of $a_{i,j} = a_i a_{i+1} \dots a_{j-1} a_j$ in $w = u_i a_i u_{i+1} \dots u_j a_j u_{j+1}$ is

function $\omega_T : \left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s \rightarrow \mathbb{N}$ (if the empty word is included in the domain and defined to have

weight 0) defined as the sum of positions of the letters of $\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s$ in w , i.e $\omega_T \left(\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s \right) = \sum_{l=i}^j {}^s I_l$.

Definition

Letting i, j varying over all indexes but with subwords of equal length.

The minimal weight of positions is $\min(\omega_T \left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s) = \frac{|a_{i,j}| \cdot (|a_{i,j}| + 1)}{2} = \frac{(j - i + 1)(j - i + 2)}{2}$ and

the maximal weight of positions given an arbitrary subword $a_{i,j}$ of w is

$$\max(\omega_T \left(\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s \right)) = |a_{i,j}| (|w| - (|w| - |a_{i,j}| + 1)) = (j - i + 1)(2|w| - j + i).$$

The *timing center* is defined as

$$C_{i,j}^{|w|} = \left\{ \left\lfloor \frac{\min(\omega_T(w_{a_{i,j}})) + \max(\omega_T(w_{a_{i,j}}))}{2} \right\rfloor, \left\lceil \frac{\min(\omega_T(w_{a_{i,j}})) + \max(\omega_T(w_{a_{i,j}}))}{2} \right\rceil \right\}.$$

Definition The *timing balance* of $\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s$ is defined as the value of a function

$$b_{\omega_T} : S_{|w|}(M) \rightarrow \mathbb{Z}, \text{ such that } b_{\omega_T} \left(\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s \right) = \omega_T \left(\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s \right) - z \text{ for some } z \in C_{i,j}^{|w|}$$

Definition $\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s$ is a *timing equilibrium* if $b_{\omega_T} \left(\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s \right) = \omega_T \left(\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s \right) = z$ for some $z \in C_{i,j}^{|w|}$, or

equivalently $\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s$ is a *timing equilibrium* if $\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s \in \ker(b_{\omega_T})$, for some $z \in C_{i,j}^{|w|}$.

Definition Let $\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_{s=1}^{|w|_{a_{i,j}}}$ mean all occurrences of $a_{i,j}$ as a scattered subword of w .

$$\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_{s=1}^{|w|_{a_{i,j}}} = \left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_1, \dots, \left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_{|w|_{a_{i,j}}}, \text{ since there are } |w|_{a_{i,j}} \text{ occurrences of } a_{i,j} \text{ as a scattered}$$

subword in w . Each $\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s$ correspond to distinct positions ${}^s I = ({}^s I_i, {}^s I_{i+1}, \dots, {}^s I_j)$.

so we have $\omega_T \left(\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_{s=1}^{|w|_{a_{i,j}}} \right) = \sum_{l=1}^j {}^s I_l, 1 \leq s \leq |w|_{a_{i,j}}$. It is possible to define a partial order on the set

of $P = \left\{ \left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_{s=1}^{|w|_{a_{i,j}}} \right\} = \left\{ \left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_1, \dots, \left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_{|w|_{a_{i,j}}} \right\}$ such that P is a chain with rank function

$$\rho : P \rightarrow [|w|_{a_{i,j}}] \text{ such that } \left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_s \mapsto s.$$

One might define the weight of a non-existing scattered subword as 0 or as $\emptyset$ (avoiding it to be included in the set of a set of *timing equilibriums*).

(so). We see that $\omega_T\left(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s\right) = \omega_T\left(\begin{bmatrix} w \\ a_i \end{bmatrix}\right) \sum_{l=i}^j I_l = {}^s I_i = |\bar{s}u_i| + 1$ ($\bar{s}u_i$ a alternative notation for the factor before the first occurrence of a_i in w where a_i is the s occurring subword in $\begin{bmatrix} w \\ a_i \end{bmatrix}_{s=1}^{w|_{a_i}}$)

Definition: The occurrence subword $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s$ and $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_t$, $1 \leq s \neq t \leq |w|_{a_{i,j}}$ are *timing weight equivalent* if $\omega_T\left(\begin{bmatrix} w \\ {}^s a_{i,j} \end{bmatrix}_s\right) = \sum_{l=i}^j {}^s I_l = \sum_{l=i}^j {}^t I_l = \omega_T\left(\begin{bmatrix} w \\ {}^t a_{i,j} \end{bmatrix}_t\right)$

Lemma If occurrence subword $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s$ and $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_t$, $1 \leq s \neq t \leq |w|_{a_{i,j}}$ are timing weight equivalent then their *occurrence(scattered concatenation) disjunction* $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_{s^\vee t}$ is a *timing equilibrium*.

(some positions might overlap) $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s \vee \begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_t = \begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_{s^\vee t}$ iff their letters have no positions incommon.

(if they are *timing weight equivalent* they have the same *weight balance*. Every subword that is *timing weight equivalent* with the *center-word* (or the empty word) is a *timing equilibrium*?)

(The above lemma is false. It could be modified to say that if

$$\omega_T\left(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s\right) = \max(\omega_T\left(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}\right) - \omega_T\left(\begin{bmatrix} w \\ b_{k,l} \end{bmatrix}_t\right)), 1 \leq s \leq |a_{i,j}|, 1 \leq t \leq |b_{k,l}| \text{ then } \omega_T\left(\begin{bmatrix} w \\ f_2 g_2^b a_2^b \end{bmatrix}_1\right)$$

$$\begin{bmatrix} w \\ a_{i,j} \cdot b_{k,l} \end{bmatrix}_{s^\vee t} \text{ is a timing equilibrium of general typ.)}$$

Define musical timings two types:

Type1: general timing compares any $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s$ with any other $\begin{bmatrix} w \\ b_{k,l} \end{bmatrix}_t$ where $|a_{i,j}| \neq |b_{k,l}|$ is a possibility.

Type2: specialized timing specialises on comparing subwords of equal length $|a_{i,j}| = |b_{k,l}|$.
q-counting polynomials, entries of the q-parikh matrix encoding

Given $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s$ we can q-count the sum of positions for each occurrence of $a_{i,j}$ as a scattered

subword of w by $P_{w, a_{i,j}}(q) = \prod_{l=i}^j q^{s_{I_l}} = q^{\sum_{l=i}^j s_{I_l}} = q^{\omega_T(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s)}$. If we count all $|w|_{a_{i,j}}$ occurrences of

$\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s$, we get $P_{w, a_{i,j}}(q) = \sum_{s=1}^{|w|_{a_{i,j}}} (\prod_{l=i}^j q^{s_{I_l}}) = \sum_{s=1}^{|w|_{a_{i,j}}} q^{\sum_{l=i}^j s_{I_l}} = \sum_{s=1}^{|w|_{a_{i,j}}} q^{\omega_T(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s)}$.

This will be the entries in the q-parikh matrix encoding and will represent the timing weight for all scattered subwords of (ascending in relation to the order of the chosen alphabet).

If we are interested in the timing balance we may construct a q-polynomial where we are q-counting this instead by the simple modification of the q-parikh encoding:

Since ${}^s C_{i,j}^{|w|} = C_{i,j}^{|w|} \quad \forall s$

Let $P_{b_{\omega_T}, w, a_{i,j}}(q)$ be the **Yamba-Parikh timing balance q-polynomial** defined as follows:

$$P_{b_{\omega_T}, w, a_{i,j}}(q) = \frac{\sum_{s=1}^{|w|_{a_{i,j}}} (\prod_{l=i}^j q^{s_{I_l}})}{\prod_{s=1}^{|w|_{a_{i,j}}} q^{s_{C_{i,j}^{|w|}}}} = \frac{\sum_{s=1}^{|w|_{a_{i,j}}} q^{\sum_{l=i}^j s_{I_l}}}{q^{\sum_{s=1}^{|w|_{a_{i,j}}} s_{C_{i,j}^{|w|}}}} = \frac{\sum_{s=1}^{|w|_{a_{i,j}}} q^{\omega_T(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s)}}{q^{|w|_{a_{i,j}} C_{i,j}^{|w|}}} = \sum_{s=1}^{|w|_{a_{i,j}}} q^{\omega_T(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s) - s_{C_{i,j}^{|w|}}} = \sum_{s=1}^{|w|_{a_{i,j}}} q^{b_{\omega_T}(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s)}$$

Example Given $\Sigma = \{c_2 < d_2 < e_2^b < f_2 < g_2^b < a_2^b\}$ and $w = d_2^2 c_2 f_2 g_2^b e_2^b d_2 c_2 f_2 g_2^b a_2^b$ (bar 4-5 in “a spring dream”). If $i=4, j=6$, then $a_{i,j} = f_2 g_2^b a_2^b$, with $|w|_{f_2 g_2^b a_2^b} = 3$. We have

$$\min(\omega_T(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s)) = \frac{(j-i+1) \cdot (j-i+2)}{2} = \frac{(3) \cdot (4)}{2} = 6 \text{ and}$$

$$\max(\omega_T(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s)) = (j-i+1) \cdot (2|w| - j + i) = (3) \cdot (2 \cdot 11 - 6 + 4) = 60. \text{ So the timing center of}$$

$$\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s \text{ is } C_{i,j}^{|w|} = \left\{ \left\lfloor \frac{6+60}{2} \right\rfloor, \left\lceil \frac{6+60}{2} \right\rceil \right\} = \{33\} \text{ (notation should be updated to } C_{|a_{i,j}|}^{|w|} \text{)}$$

Writing the indexes and the positions of $a_{i,j}$ as a scattered subword of w for each

appearance (in other words down $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_{s=1}^{|w|_{a_{i,j}}}$) we get

index	1	2	3	4	5	6	7	8	9	10	11	positions	$\omega_T\left(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s\right)$	q^{ω_T}
$\begin{bmatrix} w \\ f_2 g_2^b a_2^b \end{bmatrix}_1$	d_2	d_2	c_2	$\underline{f_2}$	$\underline{g_2^b}$	e_2^b	d_2	c_2	f_2	$\underline{g_2^b}$	$\underline{a_2^b}$	4,5,11	20	q^{20}
$\begin{bmatrix} w \\ f_2 g_2^b a_2^b \end{bmatrix}_2$	d_2	d_2	c_2	$\underline{f_2}$	g_2^b	e_2^b	d_2	c_2	f_2	$\underline{g_2^b}$	$\underline{a_2^b}$	4,10,11	25	q^{25}
$\begin{bmatrix} w \\ f_2 g_2^b a_2^b \end{bmatrix}_3$	d_2	d_2	c_2	f_2	g_2^b	e_2^b	d_2	c_2	$\underline{f_2}$	$\underline{g_2^b}$	$\underline{a_2^b}$	9,10,11	30	q^{30}

Consequently

$$P_{w,a_{i,j}}(q) = \sum_{s=1}^{|w|_{a_{i,j}}} \left(\prod_{l=1}^j q^{s I_l} \right) = \sum_{s=1}^{|w|_{a_{i,j}}} q^{\sum_{l=1}^j s I_l} = \sum_{s=1}^{|w|_{a_{i,j}}} q^{\omega_T\left(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s\right)} = \sum_{s=1}^3 \left(\left(\prod_{l=4}^6 q^{s I_l} \right) \right) = \left(\prod_{l=4}^6 q^{1 I_l} \right) + \left(\prod_{l=4}^6 q^{2 I_l} \right) + \left(\prod_{l=4}^6 q^{3 I_l} \right) = (q^4 q^5 q^{11}) + (q^4 q^{10} q^{11}) + (q^9 q^{10} q^{11}) = q^{4+5+11} + q^{4+10+11} + q^{9+10+11} = q^{20} + q^{25} + q^{30}.$$

Note that if coefficients larger than one in $P_{w,a_{i,j}}(q) = \sum_{s=1}^{|a_{i,j}|} q^{\omega_T\left(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s\right)}$ for cq^k indicate that there are

at least two timing weight equivalent $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s$ and $\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_t$. Ofcourse if they are timing weight

equivalent they are also *timing balance equivalent* since weight timing and balance timing are equivalent (proof this and you proof that the Parikh q-matrix encoding and the Yamba parikh q-matrix encoding are equivalent) (Yamba parikh q-matrix encoding of timing balance).

Proposition: $P_{w,a_{i,j}}(1) = |w|_{a_{i,j}}.$

Proof $P_{w,a_{i,j}}(1) = \sum_{s=1}^{|w|_{a_{i,j}}} 1^{\omega_T\left(\begin{bmatrix} w \\ a_{i,j} \end{bmatrix}_s\right)} = \sum_{s=1}^{|w|_{a_{i,j}}} 1 = |w|_{a_{i,j}}. \square$

Hence, the polynomials $P_{w,a_{i,j}}(q)$ “q-count” the number of occurences of $a_{i,j}$ as a scattered subword of w .

Parikh q-matrix encoding

Definition Let $M_k(q)$ denote the set of k-dimensional upper-triangular matices with entries in $\mathbb{N}(q)$ and let I_k denote the identity matrix of dimension k. The parikh q-matrix is $\overline{\psi}(a_i, j)$, $a_i \in \Sigma = \{a_1 < \dots < a_k\}$, $j \in \mathbb{N}$ defined be taking I_{k+1} and changing entry $(l, l+1)$ to q^j . Thus we have the matrix $\overline{\psi}(a_i, j) = (m_{i,j(1 \leq l \leq k+1)})$, where $m_{i,i} = 1$, $1 \leq i \leq k+1$, $m_{i,l+1} = q^j$ and all other entries $m_{i,j} = 0$.

Note that $m_{l,l+1}=1$ for $\psi(a_l)$ whereas $m_{l,l+1}=q^j$ for $\bar{\psi}(a_l, j)$. When $q=1$ we get $\bar{\psi}(a_l, j)=\psi(a_l)$.

Definition The *parikh matrix encoding* $\bar{\psi}$ is a mapping $\bar{\psi} : \Sigma^* \rightarrow M_{k+1}(q)$ defined by setting $\bar{\psi}(\lambda) = I_{k+1}$, where λ is the empty word (melodic vector) and $\bar{\psi}(w_1 w_2 \cdots w_n) = \bar{\psi}(w_1, 1) \bar{\psi}(w_2, 2) \cdots \bar{\psi}(w_n, n)$, $w_i \in \Sigma$, $1 \leq i \leq n$.

Note that $\bar{\psi}$ is not a morphism for arbitrary q . Given $\Sigma = \{a < b\}$ we have $\bar{\psi}(ab) = \bar{\psi}(a, 1) \bar{\psi}(b, 2) = \bar{\psi}(a, 1) \bar{\psi}(b, 1) = \bar{\psi}(a) \bar{\psi}(b)$.

Example Given $\Sigma = \{c_2 < d_2 < e_2^b < f_2 < g_2^b < a_2^b\}$ and $w = d_2^2 c_2 f_2 g_2^b e_2^b$ (bar 4 in “a spring dream”) $\bar{\psi}(w) = \bar{\psi}(d_2^2 c_2 f_2 g_2^b e_2^b) = \bar{\psi}(d_2, 1) \bar{\psi}(d_2, 2) \bar{\psi}(c_2, 3) \bar{\psi}(f_2, 4) \bar{\psi}(g_2^b, 5) \bar{\psi}(e_2^b, 6)$

Thus

$$\bar{\psi}(d_2^2 c_2 f_2 g_2^b e_2^b) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & q^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & q^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & q^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & q^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & q^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & q^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & q^1 + q^2 & q^7 + q^8 & q^{16} + q^{17} & 0 & 0 \\ 0 & 0 & 1 & q^6 & q^{15} & q^{25} & 0 \\ 0 & 0 & 0 & 1 & q^4 & q^9 & q^{20} + q^{25} + q^{30} \\ 0 & 0 & 0 & 0 & 1 & q^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$m_{i,j} = P_{w, a_{i,j}}(q) = P_{\left[\begin{smallmatrix} w \\ a_{i,j} \end{smallmatrix} \right]_{s=1}^{|w|_{a_{i,j}}}}(q)$$

and for $w = d_2^2 c_2 f_2 g_2^b e_2^b d_2 c_2 f_2 g_2^b a_2^b$. We get $\overline{\psi}(w) = \overline{\psi}(d_2^2 c_2 f_2 g_2^b e_2^b d_2 c_2 f_2 g_2^b a_2^b) =$

$$\begin{bmatrix} 1 & q^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & q^1 + q^2 & q^7 + q^8 & 0 & 0 & 0 \\ 0 & 0 & 1 & q^6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & q^4 & q^9 & 0 \\ 0 & 0 & 0 & 0 & 1 & q^5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \overline{\psi}(d_2, 7) \overline{\psi}(c_2, 8) \overline{\psi}(f_2, 9) \overline{\psi}(g_2^b, 10) \overline{\psi}(a_2^b, 11) =$$

$$\begin{bmatrix} 1 & q^3 & q^{11} & 0 & 0 & 0 & 0 \\ 0 & 1 & q^1 + q^2 & q^7 + q^8 & q^{16} + q^{17} & q^{26} + q^{27} & q^{37} + q^{38} \\ 0 & 0 & 1 & q^6 & q^{15} & q^{25} & q^{36} \\ 0 & 0 & 0 & 1 & q^4 & q^9 + q^{14} + q^{19} & q^{20} + q^{25} + q^{30} \\ 0 & 0 & 0 & 0 & 1 & q^5 & q^{16} + q^{21} \\ 0 & 0 & 0 & 0 & 0 & 1 & q^{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Proposition $P_{w, a_{i,j}}(q) = q^{\omega_r \left(\begin{bmatrix} w \\ a_j \end{bmatrix} \right)} \cdot P_{w, a_{i,j-1}}(q).$

Theorem (q-analog of main parikh matrix theorem)

Theorem (q-analog of theorem regarding parikh vectors of 1:st order correspond to second diagonal in parikh matrix)

Theorem (q-analog of theorem that the inverse of the parikh matrix of w is the same as the alternate matrix of the mirror word of w)

The q-parikh matrix encoding used here is injective.

(Imaginary number as solutions to real coefficient equations $\leftrightarrow$ inverse parikh matrices with negative entries as solutions to “parikh matrix equations”).)

Musical examples appendix

(Figure “the infinite piano”)

Some thoughts and speculations

When we are looking at musical compositions we can give a graphical representation of this by writing down all of its pitches when they occur. Sequence of pitches occurring in the composition are called melodies.

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